

A brief history of algebraic logic, from neat embeddings to games theory and rainbow constructions

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Abstract . We take a long magical tour in algebraic logic, starting from classical results on neat embeddings due to Henkin, Monk and Tarski, all the way to recent results in algebraic logic using so-called rainbow constructions invented by Hirsch and Hodkinson. Highlighting the connections with graph theory, model theory, and finite combinatorics, this article aspires to present topics of broad interest in a way that is hopefully accessible to a large audience. The paper has a survey character but it contains new approaches to old ones. We aspire to make our survey fairly comprehensive, at least in so far as Tarskian algebraic logic, specifically, the theory of cylindric algebras, is concerned. Other topics, such as abstract algebraic logic, modal logic and the so-called (central) finitizability problem in algebraic logic will be dealt with; the last in some detail. Rainbow constructions are used to solve problems addressing classes of cylindric-like algebras consisting of algebras having a neat embedding property. The hitherto obtained results generalize seminal results of Hirsch and Hodkinson on non-atom canonicity, non-first order definability and non-finite axiomatizability, proved for classes of representable cylindric algebras of finite dimension > 2 . We show that such results remain valid for cylindric algebras possessing relativized *clique guarded* representations that are *only locally* well behaved. The paper is written in a way that makes it accessible to non-specialists curious about the state of the art in Tarskian algebraic logic. Reaching the boundaries of current research, the paper also aspires to be informative to the practitioner, and even more, stimulates her/him to carry on further research in main stream algebraic logic.

1 Introduction

1.1 Algebraic Logic

The topic of this paper is algebraic logic in a broad sense. Initiated at the beginning of the 20th century, formal logic and its study using mathematical machinery known as metamathematical investigations, or simply metamathematics, is a crucial addition to the collection of mathematical catalysts. Traced back to the works of Frege, Hilbert, Russell, Tarski, Gödel and others, one of the branches of pure mathematics that metamathematics has precipitated is algebraic logic. This paper aspires to give a compact, yet fairly comprehensive survey, of the history of the subject up to the present day.

Algebraic logic starts from certain special logical considerations, abstracts from them, places them in a general algebraic context and via this generalization makes contact with other branches of mathematics (like set theory and topology). It can neither be maintained nor overemphasized that algebraic logic is more algebra than logic, nor that it is more logic than algebra; in this paper we argue that algebraic logic, particularly the

theory of cylindric algebras, has become sufficiently interesting and deep to acquire a distinguished status among other subdisciplines of mathematical logic, and indeed of pure mathematics. In algebraic logic a deeper understanding of both (universal) algebra and logic is not hindered by the largely irrelevant details of a particular logical system; it is instead guided structurally by clear-cut causality. Hypotheses are kept as general as possible and introduced on a by-need basis, and thus results and proofs are modular and easy to track down. Access to highly non-trivial results is also considerably facilitated.

The continuous interplay between the specific and the general in algebraic logic brings a large array of new results for particular non-conventional approaches, unifies several known results, produces new results in well-studied conventional areas, and finally reveals previously unknown causality relations. Algebraic logic can be viewed as a playground where several actors, most notably model theory, set theory, finite combinatorics and graph theory play. It is an interesting play which brought and is likely to bring more significant changes and development to the actors, and in a way, it also revolutionized our way of thinking about logic. *Algebraizing predicate logic*, a task primarily initiated by Tarski proved an extremely rewarding task, with significant early contributions from the (also) Polish logicians Rasiowa and Sikorski, as well as from Halmos who invented the theory of polyadic algebras which is a ‘cousin’ to cylindric algebras.

The history of logic, evolving into *mathematical logic or metamathematics* has been long and winding. From the beginning of the contemporary era of logic there were two approaches to the subject, one centered on the notion of logical equivalence and the other, reinforced by Hilbert’s work on metamathematics, centered on the notions of assertion and inference. It was not until much later that logicians started to think about connections between these two ways of looking at logic. Tarski [42] gave the precise connection between Boolean algebra and the classical propositional calculus. His approach builds on Lindenbaum’s idea of viewing the set of formulas as an algebra with operations induced by the logical connectives. Logical equivalence is a congruence relation on the formula algebra. This is the so-called *Lindenbaum–Tarski* method. When Tarski applied this method to the predicate calculus, it led him naturally to the concept of cylindric algebras.

Henkin began working with Tarski on the subject of cylindric algebras in the 1950’s, and a report of their joint research appeared in 1961. By then Monk had also made substantial contributions to the theory. The three planned to publish a comprehensive two-volume treatise on the theory of cylindric algebras. The first volume treated cylindric algebras from a general algebraic point of view, while the second volume contained other topics, such as the representation theory, to which Andréka and Németi contributed a great deal, and connections between cylindric-like algebras and logic. The second volume though entitled ‘Cylindric Algebras, Part 2’ also dealt with other algebraic formalisms of first order logic like Halmos’ polyadic algebras and relation algebras.

We can find that the theory of cylindric-like algebras is explicated primarily in three substantial monographs: Henkin, Monk and Tarski [13, 14], and Henkin, Monk, Tarski, Andréka and Németi [15].

This covers the development of the subject till the mid-eighties of the last century. The recent [6], referred to as ‘Cylindric Algebras, Part 3’, the notation of which we follow here, gives a representative picture of the research in the area over the last thirty years, emphasizing the bridges that cylindric algebra theory—in the wide sense—has built with other areas, like combinatorics, graph theory, data base theory, stochastics and other fields.

Not confined to the walls of pure mathematics, algebraic logic has also found bridges to such apparently remote areas as general relativity, and hyper-computation. Techniques from [19] are also used but our treatment will be self-contained; all basic notions and terminology will be recalled in due time.

This paper also surveys, refines and adds to the latest developments of the subject, but on a smaller scale, of course. In particular, we focus more on ‘pure’ Tarskian algebraic logic. We emphasize that the paper is not only purely expository.

The paper gives a survey of new recent results with fairly complete proofs, and it contains new ideas as well as new approaches to old ones. The results throughout the whole paper are not presented in a chronological order as they appeared historically; rather we move back and forth in time between deep results in algebraic logic. The temporal (historical) development of a certain topic (particularly in mathematics) does not necessarily coincide with the most logical one, for newly discovered results often shed light on older ones, resulting in a deeper understanding of both. In (algebraic) logic new paradigms usually present themselves in the form of new systems competing with old ones. This suggests a fresh look at existing logical systems, rather than their speedy overthrow.

Both developments do not even necessarily coincide with the simplest, which we aspire to achieve in this paper.

For example, take the following three branches in algebraic logic: (1) algebraization of logical calculi (deductive systems) leading up to abstract algebraic logic, (2) algebraic approach to first order logic largely due to Tarski, manifesting itself in the study of cylindric algebras (the central topic of the present paper), and finally (3) algebraic approach to non-classical propositional logics like intuitionistic and modal logic.

The work of Gabbay and Maksimova on amalgamation and interpolation categorized in (3), though historically originate in connection with the theory of polyadic and cylindric algebras through the work of pioneers including Diagneault, Johnson, Pigozzi and Comer investigating such properties for concrete classes of cylindric and polyadic algebras categorized in (2), can be classified today as the part of abstract algebraic logic dealing with definability issues categorized in (1).

The purpose of this paper is twofold. Apart from giving a general overview to some of the fundamental ideas and methods of applying algebraic machinery to logic, we also intend to present recent developments from algebraic logic and logic in an integrated format that is accessible to the non-specialist and informative for the practitioner. Our intention is to unify, illuminate and generalize several existing results scattered in the literature, hopefully stimulating further research. We hope that after reading this paper the reader will get more than a glimpse of rapid dissemination of the latest research in the field, and also contribute to it if she/he wishes.

However, algebraic logic today is a huge subject. Therefore our treatment generally is bound to be selective, focused on some but not all topics. The topics we focus on are *mainstream Tarskian algebraic logic*. Occasionally (but not always) few details can be omitted from long proofs. Our excuse will be that we wish to emphasize certain concepts while saying no more than absolutely necessary. To make up for all this, we include many references for those who wish to dig deeper. We focus on *cylindric algebras*. Notation used is common or /and self-explanatory; if not, then it will be explained at its first occurrence. Our notation is consistent with the notation in [6, 13, 14]. In particular, for an ordinal α , \mathbf{CA}_α denotes the class of cylindric algebras of dimension α . Although the

paper addresses a variety of topics in the textbooks [13, 14, 19], we require only familiarity with the minimal basics of cylindric algebra theory. In this respect, the paper is fairly self-contained. All required definitions and terminology will be recalled in due time whenever needed. Throughout the paper we make the following convention. We denote infinite ordinals by $\alpha, \beta \dots$ and finite ordinals by $n, m \dots$. Ordinals which are arbitrary meaning that they could be finite or infinite will be denoted by $\alpha, \beta \dots$. Also algebras will be denoted by Gothic letters, and when we write \mathfrak{A} for an algebra, then we shall be tacitly assuming that A denotes its universe, that is $\mathfrak{A} = \langle A, f_i^{\mathfrak{A}} \rangle_{i \in I}$ where I is a non-empty set and f_i ($i \in I$) are the operations in the signature of \mathfrak{A} interpreted via $f_i^{\mathfrak{A}}$ in \mathfrak{A} . For better readability, we omit the superscript \mathfrak{A} and we write simply $\mathfrak{A} = \langle A, f_i \rangle_{i \in I}$.

The paper is divided into two parts. The first part is a general survey on recent results in algebraic logic mostly in the last three decades building (possibly new) bridges between them. The second part is more technical addressing rainbow constructions. Such constructions are used to solve problems on classes of algebras whose members have a neat embedding property. The main theorem in the second part is theorem 9.4.

Part 1

2 Cylindric algebras

A *cylindric algebra* consists of a Boolean algebra endowed with an additional structure consisting of distinguished elements and operations, satisfying a certain system of equations. The introduction and study of these algebras has its motivation in two parts of mathematics: the deductive systems of first-order logic, and a portion of elementary set theory dealing with spaces of various dimensions, better known as cylindric set algebras; such algebras also have a geometric twist, reflected in the terminology ‘cylinder’. If we are working in 3 dimensions, and we apply the unary operations of cylindrifiers (algebraizing existential quantifiers) to a ‘circle’, then we are forming the cylinder based on this circle.

Cylindric set algebras are algebras whose elements are relations of a certain pre-assigned arity, endowed with set-theoretic operations that utilize the form of elements of the algebra as sets of sequences. For a set V , $\mathcal{B}(V)$ denotes the Boolean set algebra $\langle \wp(V), \cup, \cap, \sim, \emptyset, V \rangle$. We will primarily deal with a set V consisting of α -ary sequences, for some ordinal α .

Let U be a set and α an ordinal; α will be the dimension of the algebra. For $s, t \in {}^\alpha U$ write $s \equiv_i t$ if $s(j) = t(j)$ for all $j \neq i$. For $X \subseteq {}^\alpha U$ and $i, j < \alpha$, let

$$C_i X = \{s \in {}^\alpha U : \exists t \in X (t \equiv_i s)\}$$

and

$$D_{ij} = \{s \in {}^\alpha U : s_i = s_j\}.$$

$\langle \mathcal{B}({}^\alpha U), C_i, D_{ij} \rangle_{i, j < \alpha}$ is called *the full cylindric set algebra of dimension α* with unit (or greatest element) ${}^\alpha U$. We follow the conventions of [14] where the cylindric operations in set algebras are denoted by capital letters, while the cylindric operations in abstract algebras are denoted by small letters. Examples of subalgebras of such set algebras arise naturally from models of first order theories. Indeed, if \mathbf{M} is a first order structure in

a first order signature L with α many variables, then one manufactures a cylindric set algebra based on M as follows. Let

$$\phi^M = \{s \in {}^\alpha M : M \models \phi[s]\},$$

(here $M \models \phi[s]$ means that s satisfies ϕ in M), then the set $\{\phi^M : \phi \in Fm^L\}$ is a cylindric set algebra of dimension α , where Fm^L denotes the set of first order formulas taken in the signature L . To see why, we have:

$$\begin{aligned}\phi^M \cap \psi^M &= (\phi \wedge \psi)^M, \\ {}^\alpha M \sim \phi^M &= (\neg \phi)^M, \\ C_i(\phi^M) &= \exists v_i \phi^M, \\ D_{ij} &= (x_i = x_j)^M.\end{aligned}$$

Cs_α denotes the class of all subalgebras of full set algebras of dimension α . Recall that CA_α stands for the class of cylindric algebras of dimension α . This last (equationally defined class) is obtained from cylindric set algebras by a process of abstraction and is defined by a *finite* schema of equations given in [13, Definition 1.1.1] that holds of course in the more concrete set algebra.

Definition 2.1. By a *cylindric algebra of dimension α* , briefly a CA_α , we mean an algebra

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_i, d_{ij} \rangle_{\kappa, \lambda < \alpha}$$

where $\langle A, +, \cdot, -, 0, 1 \rangle$ is a Boolean algebra such that $0, 1$, and d_{ij} are distinguished elements of A (for all $j, i < \alpha$), $-$ and c_i are unary operations on A (for all $i < \alpha$), $+$ and \cdot are binary operations on A , and such that the following postulates are satisfied for any $x, y \in A$ and any $i, j, \mu < \alpha$:

- (C₁) $c_i 0 = 0$,
- (C₂) $x \leq c_i x$ (i.e., $x + c_i x = c_i x$),
- (C₃) $c_i(x \cdot c_i y) = c_i x \cdot c_i y$,
- (C₄) $c_i c_j x = c_j c_i x$,
- (C₅) $d_{ii} = 1$,
- (C₆) if $i \neq j, \mu$, then $d_{j\mu} = c_i(d_{ji} \cdot d_{i\mu})$,
- (C₇) if $i \neq j$, then $c_i(d_{ij} \cdot x) \cdot c_i(d_{ij} \cdot -x) = 0$.

$\mathfrak{A} \in CA_\omega$ is *locally finite*, if the dimension set of every element $x \in A$ is finite. The dimension set of x , or Δx for short, is the set $\{i \in \omega : c_i x \neq x\}$. Locally finite algebras correspond to Tarski–Lindenbaum algebras of (first order) formulas; in such algebras the dimension set of (an equivalence class of) a formula reflects the number of (finite) set of free variables in this formula. Tarski proved that every locally finite ω -dimensional cylindric algebra is representable, i.e. isomorphic to a subdirect product of set algebra each of dimension ω .

Let Lf_ω denote the class of locally finite cylindric algebras. Let RCA_ω stand for the class of isomorphic copies of subdirect products of set algebras each of dimension ω , or briefly, the class of ω dimensional representable cylindric algebras. Then Tarski's theorem reads $\text{Lf}_\omega \subseteq \text{RCA}_\omega$. This representation theorem is non-trivial; in fact it is equivalent to Gödel's celebrated Completeness Theorem [14, §4.3].

2.1 Neat embeddings and Monk's result

Soon in the development of the subject, it transpired that the class Lf_ω , the algebraic counterpart of first order logic, had some shortcomings when treated as the sole subject of research in an autonomous algebraic theory. One such assumption is the fixed dimension ω . The other is local finiteness. The restrictive character of these two notions becomes obvious when we turn our attention to cylindric set algebras. We find that there are algebras of all dimensions, and set algebras that are not locally finite are easily constructed. For these reasons the original conception of a cylindric algebra was extended. The restriction to dimension ω and local finiteness were removed, and the class CA_α , of cylindric algebras of dimension α , where α is any ordinal, finite or transfinite, was introduced. The logic corresponding to RCA_ω , allowing infinitary predicates, is a more basic algebraizable (in the standard Blok–Pigozzi sense) formalism of $L_{\omega,\omega}$.

Three pillars in the development of the subject, and even one can say *the* three pillars in the development of the subject are Tarski's result that every locally finite cylindric algebra is representable, Henkin characterization of the variety of representable algebras of any dimension via neat embeddings, in his celebrated Neat Embedding Theorem [14, Theorem 3.2.10], and Monk's proof that the variety of representable algebras of dimension > 2 cannot be axiomatized by a finite schema [27]. Monk's result had a shattering effect on the development of the subject. Not only did it create employment for many algebraic logicians in the past, but it still has its enormous repercussions till the very present day, most likely creating employment for others in the future.

The last two results involve the central notion of neat reducts:

Definition 2.2. Let $\alpha < \beta$ be ordinals and $\mathfrak{B} \in \text{CA}_\beta$. Then the α -neat reduct of \mathfrak{B} , in symbols $\text{Nr}_\alpha \mathfrak{B}$, is the algebra obtained from \mathfrak{B} , by discarding cylindrifiers and diagonal elements whose indices are in $\beta \sim \alpha$, and restricting the universe to the set $\text{Nr}_\alpha B = \{x \in \mathfrak{B} : \{i \in \beta : c_i x \neq x\} \subseteq \alpha\}$.

Let α be any ordinal. If $\mathfrak{A} \in \text{CA}_\alpha$ and $\mathfrak{A} \subseteq \text{Nr}_\alpha \mathfrak{B}$, with $\mathfrak{B} \in \text{CA}_\beta$ ($\beta > \alpha$), then we say that \mathfrak{A} *neatly embeds* in \mathfrak{B} , and that \mathfrak{B} is a β -*dilation* of \mathfrak{A} , or simply a *dilation* of \mathfrak{A} if β is clear from context. We also say that \mathfrak{A} has *a neat embedding property* and if $\beta \geq \alpha + \omega$, we say that \mathfrak{A} has **the** neat embedding property. For $\mathbf{K} \subseteq \text{CA}_\beta$, and $\alpha < \beta$, $\text{Nr}_\alpha \mathbf{K} = \{\text{Nr}_\alpha \mathfrak{B} : \mathfrak{B} \in \mathbf{K}\} \subseteq \text{CA}_\alpha$.

One can show that $\mathfrak{A} \in \text{CA}_\alpha$ has the neat embedding property $\iff \mathfrak{A} \in \text{RCA}_\alpha \iff \mathfrak{A} \in \text{SNr}_\alpha \text{CA}_{\alpha+\omega}$, cf. [13, Theorem 2.6.35]. The last equivalence is Henkin's celebrated neat embedding theorem. For $2 < n < \omega$, what Monk proved is that for any $k \in \omega$, there is an algebra $\mathfrak{A}_k \in \text{SNr}_n \text{CA}_{n+k} \sim \text{RCA}_n$, such that the ultraproduct $\prod_{k/U} \mathfrak{A}_k / U \in \text{RCA}_n$ for any non-principal ultrafilter on U . This implies that RCA_n is not finitely axiomatizable.

If the variety of representable cylindric algebras of dimension at least three had turned out to be axiomatized by a finite schema, algebraic logic would have evolved along a significantly different path than it did in the past 45 years, or so. This would have undoubtedly

marked the end of the abstract class CA_α (α an ordinal) as a separate subject of research; after all why bother about abstract algebras, if a few nice extra axioms can lead us from those to concrete algebras consisting of genuine relations, with set theoretic operations uniformly defined over these relations. However, due to Monk's non-finitizability result, together with its improvements by various algebraic logicians (from Andr  ka to Venema) CA_α was here to stay and its "infinite distance" from RCA_α , when $\alpha > 2$, became an important research topic.

Monk's non-finite axiomatizability result marked the end of an era and the beginning of a new one.

3 Atom–canonicity, Dedekind–MacNeille completions and canonical extensions

3.1 Some history

Let us start by recalling some results from the old era; from Stone's classical representability result for Boolean algebras all the way to *canonical extensions* and *completions* of *Boolean algebras with operators* of which cylindric algebras are a special case; a topic initiated by Tarski and his student Jonsson.

In 1930 Stone proved Stone's representability theorem for Boolean algebras. Stone showed that every Boolean algebra \mathfrak{B} can be embedded into a complete and atomic Boolean set algebra, namely, the Boolean algebra of the class of all subsets of the set of ultrafilters in \mathfrak{B} .

This *canonical extension* of \mathfrak{B} was characterized topologically by Jonsson and Tarski in 1950 but they went further. They developed a theory of Boolean algebras *with operators* (similar in spirit to the theory of groups with operators), proved that every Boolean algebra with operators \mathfrak{B} can be embedded into a canonical complete and atomic Boolean algebra with operators \mathfrak{A} , having the signature as \mathfrak{B} *called the canonical extension* of \mathfrak{B} , and they established a number of preservation theorems concerning equations and universal Horn sentences that are preserved under the passage from \mathfrak{B} to \mathfrak{A} . In retrospect, the representation theorem of Jonsson and Tarski is the first completeness theorem for multimodal logic using canonical models. Jonsson and Tarski concluded that the canonical extension of every abstract relation algebra is a relation algebra and similarly the canonical extension of an abstract cylindric algebra is a cylindric algebra. However, they did not settle the question of whether e.g. the canonical extension of a *representable cylindric algebra* is again representable. This problem was settled in the affirmative sometime in the 1960s, by Monk (unpublished).

MacNeille and Tarski showed in the 1930s that every Boolean algebra has another natural complete extension - and in fact it is a minimal complete extension that is formed using Dedekind cuts, now known as the *Dedekind–MacNeille completion*. Monk developed the theory of Dedekind–MacNeille completions extensions of Boolean algebras with complete operators in analogy with the Jonsson–Tarski theory of canonical extensions of Boolean algebras with operators. In particular, Monk proved an analogous preservation theorem, and concluded that the minimal complete extension of every relation and cylindric algebra is again a cylindric algebra [13]. But, unlike the case with canonical extensions, Monk was unable to settle the question of whether the Dedekind–MacNeille

completion of a representable finite-dimensional cylindric algebra of dimension > 2 is representable or not.

Contrary to canonical extensions (and possibly expectations), this question was finally settled negatively by Hodkinson in 1997 [24] by showing that for $2 < n < \omega$, \mathbf{RCA}_n is not *atom-canonical* (to be recalled below). This readily implies that \mathbf{RCA}_n is not closed under Dedekind-MacNeille completions. This result is substantially generalized in [39] by showing that for any $2 < n < \omega$, for any $k \geq 3$ the variety $\mathbf{SNr}_n \mathbf{CA}_{n+k}$ is not atom-canonical giving Hodkinson's aforementioned result when $k = \omega$ as a special case, by observing that $\mathbf{RCA}_n = \mathbf{SNr}_n \mathbf{CA}_\omega$. The result in *op.cit* will be recalled with a complete proof in theorem 9.4 where it will be shown that for $2 < n < \omega$, there is an atomic countable and simple $\mathfrak{A} \in \mathbf{RCA}_n$, such that its Dedekind-MacNeille completion, namely, the *complex algebra of its atom structure* is outside $\mathbf{SNr}_n \mathbf{CA}_{n+3}$.

3.2 Atom structures and complex algebras

We recall the notions of *atom structures* and *complex algebra* in the framework of Boolean algebras with operators of which CAs are a special case [19, Definition 2.62, 2.65]. The action of the non-Boolean operators in a completely additive (where operators distribute over arbitrary joins componentwise) atomic Boolean algebra with operators, (BAO) for short, is determined by their behavior over the atoms, and this in turn is encoded by the *atom structure* of the algebra.

Definition 3.1. (Atom Structure) Let $\mathfrak{A} = \langle A, +, -, 0, 1, \Omega_i : i \in I \rangle$ be an atomic BAO with non-Boolean operators $\Omega_i : i \in I$. Let the rank of Ω_i be ρ_i . The *atom structure* $\text{At}\mathfrak{A}$ of \mathfrak{A} is a relational structure

$$\langle \text{At}\mathfrak{A}, R_{\Omega_i} : i \in I \rangle$$

where $\text{At}\mathfrak{A}$ is the set of atoms of \mathfrak{A} and R_{Ω_i} is a $(\rho(i) + 1)$ -ary relation over $\text{At}\mathfrak{A}$ defined by

$$R_{\Omega_i}(a_0, \dots, a_{\rho(i)}) \iff \Omega_i(a_1, \dots, a_{\rho(i)}) \geq a_0.$$

Definition 3.2. (Complex algebra) Conversely, if we are given an arbitrary first order structure $\mathcal{S} = \langle S, r_i : i \in I \rangle$ where r_i is a $(\rho(i) + 1)$ -ary relation over S , called an *atom structure*, we can define its *complex algebra*

$$\mathfrak{Cm}(\mathcal{S}) = \langle \wp(S), \cup, \setminus, \phi, S, \Omega_i \rangle_{i \in I},$$

where $\wp(S)$ is the power set of S , and Ω_i is the $\rho(i)$ -ary operator defined by

$$\Omega_i(X_1, \dots, X_{\rho(i)}) = \{s \in S : \exists s_1 \in X_1 \dots \exists s_{\rho(i)} \in X_{\rho(i)}, r_i(s, s_1, \dots, s_{\rho(i)})\},$$

for each $X_1, \dots, X_{\rho(i)} \in \wp(S)$.

It is easy to check that, up to isomorphism, $\text{At}(\mathfrak{Cm}(\mathcal{S})) \cong \mathcal{S}$ alway. If \mathfrak{A} is finite then of course $\mathfrak{A} \cong \mathfrak{Cm}(\text{At}\mathfrak{A})$. For algebras \mathfrak{A} and \mathfrak{B} having the same signature expanding that of Boolean algebras, we say that \mathfrak{A} is dense in \mathfrak{B} if $\mathfrak{A} \subseteq \mathfrak{B}$ and for all non-zero $b \in \mathfrak{B}$, there is a non-zero $a \in \mathfrak{A}$ such that $a \leq b$.

An atom structure will be denoted by \mathbf{At} . An atom structure \mathbf{At} has the signature of \mathbf{CA}_α , α an ordinal, if $\mathfrak{Cm}\mathbf{At}$ has the signature of \mathbf{CA}_α , in which case we say that \mathbf{At} is an α -dimensional atom structure.

Definition 3.3. (a) Let V be a variety of \mathbf{CA}_n s. Then V is *atom-canonical* if whenever $\mathfrak{A} \in V$ and \mathfrak{A} is atomic, then $\mathfrak{CmAt}\mathfrak{A} \in V$.

(b) The *Dedekind–MacNeille completion* of $\mathfrak{A} \in \mathbf{CA}_n$, is the unique (up to isomorphisms that fix \mathfrak{A} pointwise) complete $\mathfrak{B} \in \mathbf{CA}_n$ such that $\mathfrak{A} \subseteq \mathfrak{B}$ and \mathfrak{A} is *dense* in \mathfrak{B} .

If $\mathfrak{A} \in \mathbf{CA}_n$ is atomic, then $\mathfrak{CmAt}\mathfrak{A}$ is the *Dedekind–MacNeille completion* of \mathfrak{A} . If $\mathfrak{A} \in \mathbf{CA}_n$, then its atom structure will be denoted by $\mathbf{At}\mathfrak{A}$ with domain the set of atoms of \mathfrak{A} denoted by $At\mathfrak{A}$. By the same token for a class $\mathbf{L} \subseteq \mathbf{CA}_n$ consisting of atomic algebras, we denote the class of first order structures $\mathbf{K} = \{\mathbf{At}\mathfrak{A} : \mathfrak{A} \in \mathbf{L}\}$ by $\mathbf{At}(\mathbf{L})$, so that if $\mathbf{At} \in \mathbf{K}$, then there exists $\mathfrak{A} \in \mathbf{L}$, such that $\mathbf{At}\mathfrak{A} = \mathbf{At}$.

We deal only with atom structure having the signature of \mathbf{CA}_n . *Non atom-canonicity* can be proved by finding *weakly representable atom structures* that are not *strongly representable*.

Definition 3.4. Let α be an ordinal. An atom structure \mathbf{At} of dimension α is *weakly representable* if there is an atomic $\mathfrak{A} \in \mathbf{RCA}_\alpha$ such that $\mathbf{At}\mathfrak{A} = \mathbf{At}$. The atom structure \mathbf{At} is *strongly representable* if for all $\mathfrak{A} \in \mathbf{CA}_\alpha$, $\mathbf{At}\mathfrak{A} = \mathbf{At} \implies \mathfrak{A} \in \mathbf{RCA}_\alpha$.

It can be proved for \mathbf{CA}_α s, where α is any ordinal, that $\mathbf{At}\mathfrak{A}$ is weakly representable \iff the *term algebra*, in symbols $\mathfrak{TmAt}\mathfrak{A}$ is representable. $\mathfrak{TmAt}\mathfrak{A}$ is the subalgebra of $\mathfrak{CmAt}\mathfrak{A}$ generated by the atoms. On the other hand, $\mathbf{At}\mathfrak{A}$ is strongly representable $\iff \mathfrak{CmAt}\mathfrak{A}$ is representable. Fix $2 < n < \omega$. These two notions (strong and weak representability) do not coincide for cylindric algebras as proved by Hodkinson [24]. In theorem 9.4, we generalize Hodkinson’s result by showing that there are two atomic \mathbf{CA}_n s sharing the same atom structure, one is representable and the other is even outside $\mathbf{SNr}_n \mathbf{CA}_{n+3} (\supsetneq \mathbf{RCA}_n)$. In particular, there is an algebra outside $\mathbf{SNr}_n \mathbf{CA}_{n+3}$ having a dense representable subalgebra.

4 Monk algebras, the good and the bad

Graphs will be frequently used in what follows. We recall some basics. A (*directed*) *graph* is a set G (of *nodes* or *vertices*) endowed with a binary relation E , the edge relation. A pair (x, y) of elements of G is said to be an *edge* if xEy holds. A directed graph is said to be *complete* if (x, y) is an edge for all nodes x, y . A graph is said to be *undirected* if E is symmetric and irreflexive. An undirected graph is *complete* if (x, y) is an edge for all *distinct* nodes x, y . Finite ordinals can (and will) be viewed as complete irreflexive graphs the obvious way, cf. theorem 9.4.

A *clique* in an undirected graph with set of nodes G is a set $C \subseteq G$ such that each pair of distinct nodes of C is an edge.

Definition 4.1. Let $\mathfrak{G} = (G, E)$ be an undirected graph (G is the set of vertices and E is an irreflexive symmetric binary relation on E), and C be a non-empty set of ‘colours’.

1. A subset $X \subseteq G$ is said to be an *independent set* if $(x, y) \in E$ for all $x, y \in X$.
2. A function $f : G \rightarrow C$ is called a *C colouring* of \mathfrak{G} if $(v, w) \in E$ implies that $f(v) \neq f(w)$.

3. The *chromatic number* of \mathfrak{G} , denoted by $\chi(\mathfrak{G})$, is the size of the smallest finite set C such that there exists a C colouring of \mathfrak{G} , if such a C exists, otherwise $\chi(\mathfrak{G}) = \infty$.
4. A *cycle* in \mathfrak{G} is a finite sequence $\mu = (v_0, \dots, v_{k-1})$ (some $k \in \omega$) of distinct nodes, such that $(v_0, v_1), \dots, (v_{k-2}, v_{k-1}), (v_{k-1}, v_0) \in E$. The *length* of such a cycle is k .
5. The *girth* of \mathfrak{G} , denoted by $g(\mathfrak{G})$, is the length of the shortest cycle in \mathfrak{G} if \mathfrak{G} contains cycles, and $g(\mathfrak{G}) = \infty$ otherwise.

4.1 From good to bad

Fix $2 < n < \omega$. Because RCA_n is a variety, an atomic algebra $\mathfrak{A} \in \text{RCA}_n \iff$ all equations axiomatizing RCA_n holds in \mathfrak{A} . From the point of view of $\text{At}\mathfrak{A}$, the atom structure of \mathfrak{A} , each equation in the signature of RCA_n corresponds to a certain universal monadic second-order statement, in the signature of $\text{At}\mathfrak{A}$ where the universal quantifiers are restricted to ranging over the set of atoms that lie below elements of \mathfrak{A} . Such a statement fails $\iff \text{At}\mathfrak{A}$ is partitioned into finitely many \mathfrak{A} -definable sets (sets definable using atoms of \mathfrak{A} as parameters) with certain ‘bad’ properties. Call this a *bad partition*. A bad partition of a graph is a *finite colouring*, namely, a partition of its sets of nodes into finitely many independent sets. A typical Monk argument is to construct for finite dimension > 2 a sequence of non-representable algebras based on graphs with bad partitions (having finite chromatic number) converging to one that is based on a graph having *infinite chromatic number*, hence, representable. It follows immediately that the variety of representable algebras of dimension > 2 is not finitely axiomatizable.

We call the non-representable algebras in the sequence *bad* Monk algebras. Based on graphs having finite chromatic number, they are not representable as subdirect products of set algebras. This is proved by an application of Ramsey’s theorem. But these bad algebras converge to a *good* infinite Monk algebra that is based on a graph that has infinite chromatic number. This (limit) algebra is *good in the sense that it permits a representation*; it is a subdirect product of set algebras. We borrow the terminology ‘bad and good’ from [20] where such notions are applied to the *graph used in the construction of the Monk algebra*. A graph is *good* if it has infinite chromatic number, otherwise it is, as mentioned above, *bad*, that is, it has a finite colouring.

Constructing algebras based on graphs having arbitrarily large chromatic number converging to one that is based on a graph having infinite chromatic number seems a plausible thing to do, and indeed it can be done, witnessing non-finite axiomatizability of the class of representable algebras of finite dimension > 2 . Monk’s original proof [27], refined by Maddux [26] can be seen this way. Monk used *finite* bad Monk algebras converging to a good *infinite* one. The idea involved in the construction of a bad finite Monk algebra is not so hard. Such algebras are finite, hence atomic, more precisely their Boolean reducts are atomic. The atoms are given colours, and cylindrifications and diagonals are defined by stating that monochromatic triangles are inconsistent. If a Monk’s algebra has many more atoms than colours, it follows from Ramsey’s Theorem that any representation of the algebra must contain a monochromatic triangle, so the algebra is not representable.

4.2 From bad to good

Conversely, one can form an *anti-Monk ultraproduct*, of a sequence $(\mathfrak{A}_i : i \in \omega)$ of *good* infinite Monk algebras (based on graphs with infinite chromatic number) converging to an infinite bad atomic algebra \mathfrak{A} , namely, one that is based on a graph that is *only 2-colourable* [20]. This last algebra is representable, but only *weakly*. This means that its subalgebra generated by the atoms is representable, but its Dedekind–MacNeille minimal completion, which is the complex algebra of its atom structure, namely, $\mathfrak{CmAt}\mathfrak{A}$, is not. In this sense, this limit \mathfrak{A} is not *strongly* representable. But every element of the sequence \mathfrak{A}_i ($i \in \omega$) is strongly representable, in the same sense, meaning that $\mathfrak{CmAt}\mathfrak{A}_i$ is representable. The aforementioned technique of proof, showing that the class of strongly representable cylindric algebras is not elementary is due to Hirsch and Hodkinson [20].

The *ultraproduct* that Monk used in his 1969 seminal result is a ‘reverse’ to this one, and is more intuitive, since as indicated it is plausible that a sequence of graphs having arbitrarily finite chromatic numbers getting larger and larger, converges to one that has infinite chromatic number, but the ‘other way round’ is hard to visualize, let alone implement. So how did Hirsch and Hodkinson prove their result? Fix $2 < n < \omega$. Recall that a \mathbf{CA}_n atom structure \mathbf{At} *strongly representable* \iff the complex algebra \mathfrak{CmAt} is representable. So that an atomic algebra \mathfrak{A} is strongly representable (in the sense that $\mathfrak{CmAt}\mathfrak{A} \in \mathbf{RCA}_n$) \iff its atom structure is strongly representable. One shows that an atom structure is strongly representable \iff it has *no bad partition using any sets at all*. So, here, we want to find atom structures, with no bad partitions, with an ultraproduct that *does have a bad partition*.

From a graph Γ , one can create an atom structure that is strongly representable \iff the graph is good, namely, it has no finite colouring; this atom structure is denoted by $\rho(\mathfrak{J}(\Gamma))$ in [21] and its complex algebra $\mathfrak{Cm}(\rho(\Gamma))$ is denoted by $\mathfrak{M}(\Gamma)$ in [21, Proposition 3.6.8]. So the problem reduces to finding a sequence of graphs with no finite colouring, with an ultraproduct that does have a finite colouring. We want graphs of *infinite chromatic numbers*, having an ultraproduct with *finite chromatic number*. It is not obvious, *a priori*, that such graphs actually exist. And here Erdős’ probabilistic methods offer solace. Graphs like this can be found using the probabilistic methods of Erdős, for those methods render finite graphs of arbitrarily large chromatic number and girth [10, 20]. By taking disjoint unions, we obtain graphs of infinite chromatic number (no bad partitions) and arbitrarily large girth. A non-principal ultraproduct of these has no cycles, so has chromatic number 2 (bad partition), witness the proof of [21, Corollary 3.7.2].

Both constructions from ‘bad to good’ and from ‘good to bad’ will be encountered in some detail in §9.

5 Further repercussions of the seminal result of Monk back in 1969 and its refinements; two complementary paths

Monk’s seminal result proved in 1969 [27], showing that the class of representable cylindric algebras of any dimension > 2 is not finitely axiomatizable, had a shattering effect on algebraic logic, in many respects. In fact, it changed the history of the development of the subject, and inspired immensely fruitful research, that involved dozens of publications

due to many (algebraic) logicians, starting from Andr  ka [2] all the way to Venema [9]. The conclusions drawn from this result, were that either the extra non-Boolean basic operations of cylindrifiers and diagonal elements were, due to possibly a historical accident, not properly chosen, or that the notion of representability was inappropriate; for sure it was concrete enough, but perhaps this is precisely the reason; it is far *too concrete*. Research following both paths, either by changing the signature or/and altering the notion of concrete representability have been pursued ever since, with amazing success. Indeed there are two conflicting but complementary facets of such already extensive research referred to in the literature, as ‘attacking the representation problem’. One is to sharpen Monk’s result proving *negative* (non-finite axiomatizability) results, the other is try to avoid it, proving *positive* results.

In the first path one delves deeply in investigating the complexity of potential axiomatizations for existing varieties of representable algebras. Splitting techniques [2] proved efficient to show relative non-finite axiomatizability results in the following sense: Let K be a variety having signature t , and let L be a variety having signature $t' \subseteq t$, such that if $\mathfrak{A} \in K$ then the reduct of \mathfrak{A} obtained by discarding the operations in $t \sim t'$, $\mathfrak{A}|_{t'}$ for short, is in L . We say that a set of first order formulas Σ in the signature t *axiomatizes* K *over* L , if for any algebra \mathfrak{A} in the signature t whenever $\mathfrak{A} \models \Sigma$ and $\mathfrak{A}|_{t'} \in L$, then $\mathfrak{A} \in K$. This means that Σ ‘captures’ the properties of the operations in $t \sim t'$. A *relative non-finite axiomatizability result* is typically of the form: There is no set ‘of a special form’ of first order formulas satisfying a ‘finitary condition’ that axiomatizes K over L . Such special forms may be equations, or universal formulas. By finitary, we exclusively mean that Σ is finite (this makes no sense if the signature at hand is infinite), or Σ is a finite schema in the sense of Monk’s schema [27], [14, Definition 5.6.11-5.6.12], or Σ contains only finitely many variables. The last two cases apply equally well to varieties having infinite signature like RCA_ω .

A typical result of the last form is that for $2 < n < \omega$, there is no set of universal formulas containing only finitely many variables that axiomatizes the variety of representable (quasi-) polyadic equality algebras of dimension n over the variety of representable (quasi-)polyadic equality algebras of the same dimension [2], a result that can be lifted to the transfinite. We give (more than) an outline of the proof in the infinite dimensional case deferred to an appendix.

5.1 Second path

The second path is to try and sidestep such wild unruly complex axiomatizations of RCA_α for $\alpha > 2$, often referred to as *taming methods*. Those taming methods can either involve passing to (better behaved) expansions of the algebras considered, or even completely change the signature bearing in mind that the essential operations like cylindrifiers are term definable, or else change the very notion of representability involved, as long as it remains concrete enough. The borderlines are difficult to draw, we might not know what is *not* concrete enough, but we can judge that a given representability notion is satisfactory, once we have one. This type of investigations, when one seeks to find a variety of ‘representable algebras’ that is *finitely axiomatizable by a recursive set of equations*, is known in the literature as the *finitizability problem* [28, 33, 29, 39], to be dealt with below in some detail.

To get a grasp of how difficult the *representability problem for \mathbf{CA}_α s* seemed to be in the late sixties of the last century we quote Henkin, Monk and Tarski [13, pp.416] addressing the problem for the time being from the ‘algebra side’:

‘There are two outstanding open problems, one of them is the problem of providing a simple intrinsic characterization for all representable \mathbf{CA}_α s, the second problem is to find a notion of representability for which a general representation theorem could be obtained which at the same time would be close to geometrical representation in the concrete character and intuitive simplicity. It is by no means clear that a satisfactory solution of either of these problem will ever be found or that a solution is possible’! (Our exclamation mark).

Classical (Tarskian) representation: Let α be an ordinal. Recall that \mathbf{Cs}_α denotes the class of *set algebras of dimension α* , that is, algebras with top elements a *cartesian square*; a set of the form ${}^\alpha U$ for some non-empty set U . When $\alpha < \omega$, set algebras are simple, and the class \mathbf{RCA}_α is by definition \mathbf{SPCs}_α . The definition of representability, without any change in its formulation, is extended to algebras of infinite dimension.

In this case, however, an intuitive justification is less clear since cylindric set algebras of infinite dimension are not in general subdirectly indecomposable. In fact, for $\alpha \geq \omega$ no intrinsic property is known which singles out the algebras isomorphic to \mathbf{Cs}_α s among all representable \mathbf{CA}_α ’s, as opposed to the finite dimensional case where such algebras can be intrinsically characterized by the property of being simple; for $\alpha < \omega$, $\mathfrak{A} \in \mathbf{RCA}_\alpha$ is simple $\iff \mathfrak{A} \in \mathbf{ICs}_\alpha$ where \mathbf{I} denotes the operation of closing under isomorphic copies.

But in any case (even for $\alpha \geq \omega$), members of \mathbf{RCA}_α can be still represented as algebras consisting of genuine α -ary relations over a disjoint union of Cartesian squares, the class consisting of all such algebras is denoted by \mathbf{Gs}_α , with \mathbf{Gs} standing for *generalized set algebras*; these are algebras whose top elements are *disjoint unions* of cartesian squares (of dimension α).

The class of generalized cylindric set algebras, just as that of ordinary cylindric set algebras, has many features that make it well qualified to represent \mathbf{CA}_α . The construction of the algebras in this (bigger) class retains its concrete character, all the fundamental operations and distinguished elements are unambiguously defined in set-theoretic terms, and the definitions are uniform over the whole class; geometric intuition underlying the construction gives us good insight into the structures of the algebras. Thus there is (geometric) justification that \mathbf{Gs}_α consists of the standard models of \mathbf{CA} -theory. Its members consist of *genuine α -ary relations*, and the operations are *set-theoretically concretely defined* utilizing the form of these relations as sets of sequences. But contrary to what was hoped for, by the results of Andr  ka and Monk [27, 2] any axiomatization of this class of (representable) algebras has to be infinite and extremely complex if $\alpha > 2$. The axioms stipulated by Tarski were not enough to obtain a Stone like representability result. *Infinitely many* more axioms were needed. We will see later in §7, cf. theorem 7.5, that removing the condition of *disjointness* is highly rewarding, in the context of the so-called finitizability problem, where we can (and will) obtain a Stone-like representability result expressed via a finite set of equations enforcing representability. The last procedure is an instance of the technique of *relativization*.

Relativization: One can find well motivated appropriate notions of semantics (a notion of representation) by first locating them while giving up classical semantical prejudices. It is hard to give a precise mathematical underpinning to such intuitions. What really counts at the end is a completeness theorem stating a natural fit between chosen in-

tuitive concrete-enough, but perhaps not *excessively* concrete, semantics and well behaved, hopefully recursive, axiomatizations. Gödel's completeness theorem ties just one choice of predicate logical validity in 'standard set theoretic modelling'. One could be impressed by the beauty of ensnaring 'intuitive validity' by means of exact mathematical notions. But on the other hand, one can argue that from equally natural semantic points of view, other logical equilibria arise with different sets of validities.

Fix the dimension n to be finite > 1 . A technique that proved extremely potent in obtaining positive results in both algebraic and modal logic is that of *relativization*, the slogan being *relativization turns negative results positive*. The technique originated from research by Istvan Németi dedicated to the class of *relativized set algebras* whose top elements are arbitrary sets of n -ary sequences, $n < \omega$, that are not necessarily *squares* of the form nU , where U a non-empty set and $n < \omega$ is the dimension. So the top element of a relativized set algebra of dimension n is an *arbitrary subset* V of nU with operations defined like cylindric set algebras of dimension n , but relativized to V . Németi proved, in a seminal result, that the universal theory of such algebras is decidable. From the modal perspective, such top elements are called *guards* 'guarding the semantics'. The corresponding multimodal logic exhibits nice modal behaviour and is regarded as the base for proposing the so-called *guarded fragments* of first order logic by Andr  ka et al. [9]. Relativized semantics has led to many other nice modal fragments of first order logic, like the loosely guarded, clique guarded and packed fragments of first order logic [9], [19, Definitions 19.1, 19.2, 19.3, p. 586-589]. The most severe relativization is that one deals with set algebras whose top element V is an arbitrary set of n -ary sequences and operations defined like cylindric set algebra of dimension n relativized to V (dealing with the class \mathbf{CrS}_n). Less severe relativizations are obtained by imposing certain closure properties on the top element V without losing the positive properties of \mathbf{CrS}_n like e.g. the decidability of its equational theory. We say that $V \subseteq {}^nU$ is *locally square* if whenever $s \in V$ and $\tau : n \rightarrow n$, then $s \circ \tau \in V$. Let \mathbf{G}_n be the class of set algebras whose top elements are locally square and operations are defined like cylindric set algebra of dimension n relativized to the top element V , together with with the unary substitution operators denoted by $S_{[i,j]}$ ($i, j < n$), where for $X \subseteq V$, then $S_{[i,j]}X = \{s \in V : s \circ [i,j] \in V\}$, where $[i,j]$ is the transposition that swaps i and j . So \mathbf{G}_n has the same signature as *polyadic equality algebras of dimension n* .

It is proved in [11] that the class \mathbf{G}_n is a finitely axiomatizable variety obtaining a Stone-like representability result for algebras of n -ary relations. In [39] this result is reproved using games. A precursor to the result in [11] is the classical Andr  ka-Resek-Thompson result [8] which says that every n -dimensional algebra that has the same signature as \mathbf{CA}_n , satisfying a certain finite set of equations together with the so-called *merry go round identities*, briefly, **MGR** is representable by set algebras whose top elements are *diagonalizable* in the following sense: If $V (\subseteq {}^nU)$ is the top element of a given set algebra of dimension n , then $s \in V \implies s \circ [i,j] \in V$ where $[i,j]$ is the replacement that sends i to j and keeps everything else fixed. As before, the operations are like those of cylindric set algebras of dimension n relativized to V . The procedure of weakening the threatening condition of commutativity of cylindrifiers *globally*, like in the case of *several relativized set algebras including \mathbf{CrS}_n and \mathbf{G}_n* ($1 < n < \omega$) (thereby guarding semantics) proved highly rewarding. In [39] it is shown that the universal theories of such varieties is decidable. This technique of relativization also works for infinite dimensions as will be

shown in theorem 7.5.

In all cases finite and infinite dimensions, one can obtain positive results on finite axiomatizability, atom-canonicity, canonicity, and complete representations for the hitherto obtained varieties of representable (modal) algebras, cf. [39]. So far, decidability of the universal theory of the variety of representable algebras is known only for finite dimensions as shown next. Recall that $\mathfrak{B}(V)$ is the Boolean algebra $(\wp(V), \cup, \cap, \sim, \emptyset, V)$. We denote the class of set algebras of the form $(\mathfrak{B}(V), \mathbf{C}_i, \mathbf{D}_{ij})_{i,j < n}$ where V is diagonalizable by \mathbf{D}_n and that consisting of algebras of the form $(\mathfrak{B}(V), \mathbf{C}_i, \mathbf{D}_{ij}, \mathbf{s}_{[i,j]})_{i,j < n}$ where V is locally square by \mathbf{G}_n .

Theorem 5.1. [8, 11, 12, 5]. *Fix $n > 1$. Then \mathbf{D}_n and \mathbf{G}_n are varieties that are axiomatizable by a finite schemata. In case $n < \omega$, both varieties are finitely axiomatizable and have a decidable universal (hence equational) theory.*

Proof. We prove only decidability of the universal theory of \mathbf{G}_n . The proof depends on the decidability of the loosely guarded fragment of first order logic. For $\mathfrak{A} \in \mathbf{G}_n$, let $\mathfrak{L}(\mathfrak{A})$ be the first order signature consisting of an n -ary relation symbol for each element of \mathfrak{A} . Then we show that for every $\mathfrak{A} \in \mathbf{G}_n$, for any $\psi(x)$ a quantifier free formula of the signature of \mathbf{G}_n and $\bar{a} \in \mathfrak{A}$ with $|\bar{a}| = |\bar{x}|$, there is a loosely guarded $\mathfrak{L}(\mathfrak{A})$ sentence $\tau_{\mathfrak{A}}(\psi(\bar{a}))$ whose relation symbols are among \bar{a} such that for any relativized representation M of \mathfrak{A} , $\mathfrak{A} \models \psi(\bar{a}) \iff M \models \tau_{\mathfrak{A}}(\psi(\bar{a}))$.

Let $\mathfrak{A} \in \mathbf{G}_n$ and $\bar{a} \in \mathfrak{A}$. We start by the terms. Then by induction we complete the translation to quantifier free formulas. For any tuple \bar{u} of distinct n variables, and term $t(\bar{x})$ in the signature of \mathbf{G}_n , we translate $t(\bar{a})$ into a loosely guarded formula $\tau_{\mathfrak{A}}^{\bar{u}}(t(\bar{a}))$ of the first order language having signature $L(\mathfrak{A})$. If t is a variable, then $t(\bar{a})$ is a for some $a \in \text{rng}(\bar{a})$, and we let $\tau_{\mathfrak{A}}^{\bar{u}}(t(a)) = a(\bar{u})$. For \mathbf{d}_{ij} one sets $\tau_{\mathfrak{A}}^{\bar{u}}(t)$ to be $\mathbf{d}_{ij}^{\bar{u}}(\bar{u})$ and the constants 0 and 1 are handled analogously. Now assume inductively that $t(\bar{a})$ and $t'(\bar{a})$ are already translated. We suppress \bar{a} as it plays no role here. For all $i, j < n$ and $\sigma : n \rightarrow n$, define (for the clause \mathbf{c}_i , w is a new variable):

$$\begin{aligned}\tau_{\mathfrak{A}}^{\bar{u}}(-t) &= 1(\bar{u}) \wedge \neg \tau_{\mathfrak{A}}^{\bar{u}}(t), \\ \tau_{\mathfrak{A}}^{\bar{u}}(t + t') &= \tau_{\mathfrak{A}}^{\bar{u}}(t) + \tau_{\mathfrak{A}}^{\bar{u}}(t'), \\ \tau_{\mathfrak{A}}^{\bar{u}}(\mathbf{c}_i t) &= 1(\bar{u}) \wedge \exists w [1(\bar{u}_w^i) \wedge \tau_{\mathfrak{A}}^{\bar{u}_w^i}(t)], \\ \tau_{\mathfrak{A}}^{\bar{u}}(\mathbf{s}_{\sigma} t) &= 1(\bar{u}) \wedge (\tau_{\mathfrak{A}}^{\bar{u} \circ \sigma}(t)).\end{aligned}$$

Let M be a relativized representation of \mathfrak{A} , then $\mathfrak{A} \models t(\bar{a}) = t'(\bar{a}) \iff M \models \forall \bar{u} [\tau_{\mathfrak{A}}^{\bar{u}}(t(\bar{a})) \longleftrightarrow \tau_{\mathfrak{A}}^{\bar{u}}(t'(\bar{a}))]$. For terms $t(\bar{x})$ and $t'(\bar{x})$ and $\bar{a} \in \mathfrak{A}$, choose pairwise distinct variables \bar{u} , that is for $i < j < n$, $u_i \neq u_j$ and define $\tau_{\mathfrak{A}}(t(\bar{a}) = t'(\bar{a})) := \forall \bar{u} [1(\bar{u}) \rightarrow (\tau_{\mathfrak{A}}^{\bar{u}}(t(\bar{a})) \longleftrightarrow \tau_{\mathfrak{A}}^{\bar{u}}(t'(\bar{a})))]$. Now extend the definition to the Boolean operations as expected, thereby completing the translation of any quantifier free formula $\psi(\bar{a})$ in the signature of \mathbf{G}_n to the $L(\mathfrak{A})$ formula $\tau_{\mathfrak{A}}(\psi(\bar{a}))$.

Then it is easy to check that, for any quantifier free formula $\psi(\bar{x})$ in the signature of \mathbf{G}_n and $\bar{a} \in \mathfrak{A}$, we have:

$$\mathfrak{A} \models \psi(\bar{a}) \iff M \models \tau_{\mathfrak{A}}(\psi(\bar{a})),$$

and the last is a loosely guarded $\mathfrak{L}(\mathfrak{A})$ sentence. By decidability of the loosely guarded fragment the required result follows. \square

Legitimacy of relativization: A mathematical theory of ‘meaning’ leads to new insights and clarifications. Virtually all logics considered by logicians permit a semantical approach. In some cases, such as intuitionistic logic, there are philosophical reasons to prefer one semantics over another. But research in algebraic logic has shown that this is the case too with *classical and modal logic*, and furthermore, in these two cases, the motivation of altering semantics is more often than not *more basic and practical*. The move of altering semantics (the most famous is Henkin’s move changing second order semantics) has radical philosophical repercussions, taking us away from the conventional Tarskian semantics captured by Gödel–like axiomatization. The latter completeness proof is effective but highly undecidable; and this property is inherited by finite variable fragments of first order logic when the number of variables present in the signature is at least three, as long as we insist on Tarskian (square) semantics.

For higher order logics Henkin’s approach uses general models with restricted ranges of quantification. Standard models remain a limiting case where all mathematically possible sets are present. This makes higher order logic many sorted first order logic treating individual, sets and predicates a par. The move from higher order to first order brings clear gains in complexity. But though effectively axiomatizable predicate logic is undecidable by Church’s theorem. We can gain more by *relativization*. By relativization we experience immediate practical benefits, obtaining quite expressive *decidable* quantifier logics. This dynamic viewpoint enriches and unifies our view of a mutiplicity of disciplines sharing a cognitive slant. Standard predicate logic has arisen historically by making several ad–hoc semantic decisions that could well have gone in a different way. Its not all about *one canonical* completeness theorem but rather about *several* completeness theorems obtained by varying both the semantic and syntactical parameters seeking an optimal fit. This can be implemented from a classical relativized representability theory, like that adopted in the monograph [13], though such algebras were treated in *op.cit* off main stream, and they were only brought back to the front of the scene by the work of Resek, Thompson, Andréka [8] and last but not least Ferenczi [12].

Defining the notion of a representation: Inspite of the infinite discrepancy between abstract and concrete algebras, there are (other) means to control the ‘notion of a representation’. Representations of an algebra can be described in a *first order theory in a two–sorted language*. The first sort in a model of this *defining theory* is the algebra itself, while the second sort is a representation of it. The defining theory specifies the relation between the two, and its axioms depend *on what kind of representation*, be it relativized, ordinary, complete, etc, is considered.

Thus the representable algebras are those models of the first sort of the defining theory, with the second sort providing the representation.

Many classes in algebraic logic can be seen this way. The defining theory is usually finite, simple and essentially recursively enumerable; and if we are lucky it will be recursive. The class of all structures that arise as the first sort of a model of a two–sorted first order theory is an old venerable notion in model theory introduced by Maltsev in the forties of the 20th century. Ever since it was studied by Makkai and others. It is known as a pseudo–elementary class. What is meant here is a PC_{Δ} class [19] but expressed in a two sorted language. The term pseudo–elementary class strictly means PC_{Δ} when the second sort is empty, but the two notions were proved to be equivalent by Makkai [19]. Any elementary class is pseudo–elementary, but the converse is not true; the class of α –

dimensional neat reducts of β -dimensional cylindric algebras for $1 < \alpha < \beta \cap \omega$ is an example [38]. Other examples are the class of strongly representable atom structures and the class of completely representable algebras, both of finite dimension > 2 , as proved by Hirsch and Hodkinson [38, 18, 20],

According to Hirsch and Hodkinson [19] a fairly but not completely general *definition* of 'a notion of representation' is just the second sort of a model of a two-sorted (more often than not recursively enumerable) first order theory, where the first sort of the theory is the algebra. Put in this form, Hirsch and Hodkinson apply model-theoretic finite forcing to representing various abstract classes of algebras, using possibly relativized representations. Model-theoretic forcing, with precursors Henkin's proof of the classical completeness theorem for first order logic and its algebraic version; the neat embedding theorem (proved also by Henkin), typically involves constructing a model of a first order theory by a *game*.

Axiomatizing the class of representable algebras using games: Games are a highly structured activity with powerful concrete intuitions concerning moves and strategies. This view has been around as an undercurrent in logic for quite some time, emerging often in various contexts. The game builds the model with elements being produced by the second player called \exists (verifier), in his response to criticism by the first player, called \forall (falsifier). The falsifier draws objects from the domain of discourse which can be tested for certain facts. Semantical notions of validity serve as a touch-stone of adequacy for proof theoretic or game theoretic proposals, but the latter provides more vivid ideas and elaborate concepts about structuring of arguments and procedures of reasoning. The analysis of winning strategies of the players involved during the play gives fine and delicate information, and allows one to delve deeper into the analysis. The approach of Hirsch and Hodkinson is basically to combine the forcing games with the two sorted approach mentioned above to representations.

For example, games can be used to give an explicit (necessarily infinite) recursive axiomatization of the class \mathbf{RCA}_n for $2 < n < \omega$. We give a rough outline now. The idea is that a winning strategy for \exists in a k -rounded game for $k < \omega$, call it \mathbf{G}_k , can be coded in a first order sentence ρ_k in the signature of \mathbf{CA}_n .

This game is played on so-called *networks on \mathfrak{A}* . A pre-network N on \mathfrak{A} is a map $N : {}^n\Delta \rightarrow \mathfrak{A}$, where Δ is a finite set of nodes, denoted by $\mathbf{nodes}(N)$. A network is roughly a 'finite approximation to a representation' so it is a pre-network that satisfies certain consistency conditions. For example, for $\bar{x}, \bar{y} \in {}^n\Delta$, $N(\bar{x}) \leq \mathbf{d}_{ij} \iff x_i = x_j$ and $\bar{x} \equiv_i \bar{y} \implies N(\bar{x}) \cdot c_i N(\bar{y}) \neq 0$. The k -rounded game between \forall and \exists where the board of play consists of networks on \mathfrak{A} is denoted by $\mathbf{G}_k(\mathfrak{A})$. Suppose that we are at round $0 < t < \omega$ and $t < k$. Then \forall challenges \exists by choosing a previously played network N_t an edge \bar{x} of N_t an index $i < n$ and $a \in \mathfrak{A}$. \exists can either *reject this move* by playing the pre-network N_{t+1} that is like N_t except that $N_{t+1}(\bar{x}) = N_t(\bar{x}) \cdot -c_i a$. If she accepts, then \exists has to respond with a network N_{t+1} *refining* N_t ; *in this respect being a finer finite approximation to a representation*. In particular, $N_{t+1} \supseteq N_t$ (as maps). In her response, \exists plays a pre-network defined as follows: $\mathbf{nodes}(N_{t+1}) = \mathbf{nodes}(N_t)$ plus a new node z . Let \bar{z} be given by $\bar{z} \equiv_i \bar{x}$, $z_i = z$. The labels of the n -tuples (hyperedges) of $\mathbf{nodes}(N_{t+1})$ are given by

- $N_{t+1}(\bar{z}) = a \cdot \prod_{j, k < n, z_j = z_k} \mathbf{d}_{jk}$
- $N_{t+1}(\bar{x}) = N_t(\bar{x}) \cdot c_i a$

- $N_{t+1}(\bar{y}) = N_t(\bar{y})$ for all $y \in {}^n N_t \sim \{\bar{x}\}$
- $N_{t+1}(\bar{y}) = \prod_{j,k < d, y_j = y_k} d_{jk}$ for all $y \in {}^n N_{t+1} \sim (\{\bar{z}\} \cup {}^n N_t)$.

If N_{t+1} is a *network*, in this case we say that \exists survives round t . If \exists survives every round $t < k$, then \exists wins; otherwise \forall wins. There are no draws. For a countable algebra \mathfrak{A} , $\mathfrak{A} \models \rho_k \iff \exists$ has a winning strategy in $\mathbf{G}_k(\mathfrak{A})$. One can translate the set of such sentences $\{\rho_k : k \in \omega\}$ to a set of equations Σ using that \mathbf{RCA}_n is a discriminator variety. If \mathfrak{A} is countable and $\mathfrak{A} \models \Sigma$, then this means that \exists has a winning strategy (expressed by) ρ_k in $\mathbf{G}_k(\mathfrak{A})$ for all $k \in \omega$. Using a compactness argument, one shows that \exists has a winning strategy in $\mathbf{G}_\omega(\mathfrak{A})$. Then \exists can use her winning strategy in $\mathbf{G}_\omega(\mathfrak{A})$, to build a representation of \mathfrak{A} . The countability condition here is essential, so that all possible moves by \forall can be scheduled and it can be assumed that \exists has succeeded to respond to every possible move (challenge) by \forall .

\exists builds the required representation as the ‘limit’ of the play follows: Consider a play $N_0 \subseteq N_1 \subseteq \dots$ of $\mathbf{G}_\omega(\mathfrak{A})$ in which \exists uses his winning strategy and \forall plays every possible legal move. The *limit* of the play is a representation of \mathfrak{A} defined as follows: Let $N = \bigcup_{t < \omega} N_t$ and define $h : \mathfrak{A} \rightarrow \wp(N)$ as follows:

$$h(a) = \{\bar{x} \in N : \exists t < \omega (\bar{x} \in N_t \& N_t(\bar{x}) \leq a)\}.$$

One then readily deduces that $\text{Mod } \Sigma = \mathbf{RCA}_n$, for if $\mathfrak{C} \models \Sigma$, then by the downward Tarski–Löwenheim–Skolem theorem there is an elementary subalgebra of \mathfrak{C} , \mathfrak{A} say, such that $\mathfrak{A} \models \Sigma$. By the above reasoning \mathfrak{A} is representable. But \mathbf{RCA}_n is a variety, hence closed under elementary equivalence, so \mathfrak{C} is representable, too, cf. [19, §.8.3]. A concrete instance of such games will be encountered in theorem 6.4.

Representing all \mathbf{CA}_n s (after all) by twisting set algebras? To understand the “essence” of representable algebras, one often deals with the non-representable ones, the “distorted images” so to speak. Simon’s result in [41], of “representing” non-representable algebras, seems to point out that this distortion is, after all, not completely chaotic. This is similar to studying non-standard models of arithmetic, that do shed light on the standard model. From the main result in [8] (the famous Andr  ka–Resek–Thompson theorem), it follows that if $2 < n < \omega$, and $\mathfrak{A} \in \mathbf{CA}_n$ satisfies the so-called *merry go round identities*, **MGR** for short, then \mathfrak{A} is representable as a relativized set algebra having top element a set $V \subseteq {}^n U$ such that if $s \in V$, and $i, j < n$ then $s \circ [i|j] \in V$.

Let α be an ordinal ≥ 3 . Instead of asking “What is missing from \mathbf{CA} ’s to be representable?”, Henkin turned around the question and asked how much set algebras needed to be distorted to provide a representation of all \mathbf{CA}_α ’s. And, strikingly, the answer provided by Simon is “not very much”, at least for the lowest value of α , for which Monk’s seminal result and its improvements apply, namely, $\alpha = 3$. Simon [41] proved that any abstract 3-dimensional cylindric algebra satisfying **MGR** can be obtained from a \mathbf{Cs}_3 (a cylindric set algebra of dimension 3) by the methods called *twisting and dilation*, studied in [14, pp. 86–91]. This, in a way, does add to our understanding of the distance between the abstract notion of cylindric algebra and its concrete one, at least in the case of dimension 3. However, Simon had to broaden Henkin’s notion of twisting to exhaust the class \mathbf{CA}_3 . He also showed that Henkin’s more restrictive notion of twisting does not fit the bill; there are abstract \mathbf{CA}_3 ’s satisfying the **MGR** that cannot be obtained by the methods of

relativization, dilation and twisting, the latter understood in the sense of Henkin. Simon’s twisting is a stronger “distortion” of the original algebra, and so its scope is wider, it can “reach” more algebras. The analogous problem for higher dimension is still an intriguing open problem. The idea is that given any abstract $\mathfrak{A} \in \mathbf{CA}_3$ then there is a set algebra $\mathfrak{C} \in \mathbf{Cs}_3$, such that \mathfrak{A} can be obtained from \mathfrak{C} by applying a finite number of operations among twisting, dilating, relativizing and forming subalgebras, not necessarily all (e.g. if $\mathfrak{A} \in \mathbf{Cs}_3$), and not necessarily in this order.

Let \mathfrak{Ca}_α denote the class of \mathbf{CA}_α atom structures. Recall that the idea behind dilations is that in a CA if it is an atom can be inserted in a certain position, as long as its addition does not contradict the CA axioms. Twisting, originally, consists of starting from a complete atomic \mathbf{CA}_α \mathfrak{A} , selecting atoms $a, b \in \mathfrak{A}$ and an ordinal $k < \alpha$ and then redefining c_k on a and b by interchanging the action of c_k on a and b , in part, “twisting”. Twisting is used to “distort” atom structures. It produces \mathfrak{Ca}_α ’s from \mathfrak{Ca}_α ’s, and it typically kills MGR. However, in some circumstances it *can reproduce* MGR!. In both dilation and twisting, one starts out with a complete and atomic \mathbf{CA}_α s, adjoins new elements and /or changes the operations to get a new, complete atomic \mathbf{CA}_α with certain prescribed properties.

Now here is an outline of how to ‘represent’ every \mathbf{CA}_3 [41]: One starts with a \mathbf{CA}_3 \mathfrak{A} and checks if MGR holds in it. If it does then by the Andr eka–Resek–Thompson theorem [8], \mathfrak{A} is a subalgebra of a relativized \mathbf{Cs}_3 , and we are done. Else, one embeds \mathfrak{A} into its complete and atomic canonical extension \mathfrak{A}^+ in order to be able to repair the failure of MGR by twisting the atom structure of \mathfrak{A}^+ . But one has to apply dilation first by inserting atoms to adjust the twisting parameters, so that twisting can be applied in the first place. Twisting is then applied to the algebra \mathfrak{B} obtained by dilating \mathfrak{A}^+ to get a \mathbf{CA}_3 , \mathfrak{A}_3 in which MGR holds. Then one uses the Andr eka–Resek–Thompson result to represent \mathfrak{A}_3 as a relativized \mathbf{Cs}_3 . So here twisting is used in a more constructive way; by twisting an algebra in which MGR does not hold, we get one where MGR holds.

Since the effect of twisting can always be undone by twisting the twisted algebra, the procedure we have described shows that \mathfrak{A} can be obtained from a subalgebra of a relativized set algebra $\mathfrak{A}_3 \in \mathbf{CA}_3$ by applying dilation, twisting, relativization and the operation of forming subalgebras, not necessarily in this order.

6 The interaction with modal logic; non–orthodox rules

Obtaining positive representability results can also be implemented from a modal perspective, in even a more natural way, a task primarily initiated by Yde Venema in his dissertation. This approach has enriched the subject considerably, dissolving the so robust persistent non–finite axiomatizability results in algebraic logic initiated by Monk. The correspondence between modal logic and algebraic logic allows one, on a first level, to study the algebras to understand the deductive systems. But often metalogical properties end up having logical counterparts. Though for a long time the algebraic and logical strands have been carried out in relative isolation, now the tie between these two approaches is considerably strengthened. Algebraising modal logic allows strong new techniques to bear on metalogical problems. Furthermore, algebraic semantics turns out better behaved than

Kripke frame based semantics. For example there is an (algebraic) completeness theorem for every normal modal logic, dually there is a representation theorem for Boolean algebras with operators extending Stone’s representability result, due to Jónsson and Tarski. There is no analogous result for frames.

At a deeper level of this interplay, is e.g the result that is undecidable to tell whether a given finite \mathbf{CA}_n , for $2 < n < \omega$, is representable used to obtain strong undecidability results for multimodal logics between the product multimodal logics \mathbf{K}^n and $\mathbf{S5}^n$ [25]. Conversely, in his pioneering paper ‘Modal Cylindric Algebras’ [9], Venema crossed the bridge from the other side. Standard Hilbert style axiomatizations, being excluded by Monk’s result, Venema succeeded to obtain a sound and complete proof system for finite variable fragments of first order logic (disguised in a modal formalism) with at least three variables; using instead non-orthodox derivation rules involving the so-called difference operator as illustrated in some detail next.

On the one hand, one can view modal logics as fragments of first order logic. But on the other hand, one can also turn the glass around and give first order logic with n variables a modal formalism, by viewing assignments as worlds, and giving the existential quantifier the most prominent citizen in first order logic the following familiar modal pattern:

$$M, s \models \exists x_i \phi \iff (\exists s)(s \equiv_i t) \& M, s \models \phi.$$

(Recall that $s \equiv_i t$ means that s and t agree off of i , that is $t(j) = s(j)$ for all $j \neq n$). Here existential quantifiers are viewed as modalities. A representation theorem for algebras like the one mentioned in theorem 5.1 is **the dual** of representing abstract state frames by what Venema calls assignment frames (sets of sequences). In the aforementioned theorem the frames had domains diagonalizable or locally square ‘set of sequences’. The Stone-like representability result proved for such algebra in [8, 12], and later by games in [39] are in essence algebraic.

But in some other cases like in the ‘squareness case’ it might be easier to work on **the frame level**. Indeed it *is easier*. Fix $2 < n < \omega$.

Venema worked on the frame level of *squares*, namely, atom structures of the form $({}^nU, T_i, D_{ij})_{i,j < n}$, where the accessibility relation corresponding to cylindrifiers and diagonal elements are defined by: For $s, t \in {}^nU$, $sT_it \iff s \equiv_i t$ and $D_{i,j} = \{s \in {}^nU : s_i = s_j\}$. This (modal) view enabled him to use non-orthodox derivation rule common in modal logic, to get a completeness theorem for finite variable fragments of first order logic.

The completeness result achieved by Venema for what he calls *cylindric modal logic* (a dual formalism of finite variable fragments of first order logic), has its roots in the prophetic monograph [14]. Such techniques are the *frame version* of the notion of so-called *richness* [14, Definition 3.2.1]. A *rich* algebra has a rich supply of elements, that are sufficient to satisfy a certain set of equations [14, Lemma 3.2.3, Theorem 3.2.5] to be recalled in a moment. And it is precisely the (dual of the) richness condition that can be transformed into a non-orthodox derivation rule as will be shown in a moment. But such a rule, as it stands on its own, is only *sound*, but it is also ‘potentially complete’. One can add only finitely many axioms to get completeness. From the modal point of view the unorthodox completeness theorem is based on a *special characterization* of the n -squares nU (domain the Kripke frames).

The starting point for this characterization is the observation that the inequality relation on such n -squares can be obtained in as a certain composition of the accessibility

relations (corresponding to cylindrifiers and diagonal elements), using the fact that the difference operator on squares is term definable in a ‘nice’ way.

A relation R^n as a new accessibility relation is added to abstract Kripke frames (atom structures of \mathbf{CA}_n s). Its aim is to enforce representability of the complex algebra based on this frame. This relation can be modally expressible (else there would have been no problem!), but it can be reflected by an operator D_n which acts as a **difference operator**.

The crucial key idea here is that a frame is representable as a square $\iff R^n$ is an inequality relation.

This is a complicated condition, that as indicated cannot be modally expressed, but it can be ‘captured’ if we consider the *difference operator* D . This operator D increases the expressive power of modal languages. For example, it can express frame properties that are not expressible modally, such as irreflexivity via $\Diamond p \rightarrow Dp$. This definability of R^n using a ‘difference-like operator’, denote it by D_n , leads to a new non-orthodox derivation rule, proving completeness. What makes this rule non-orthodox, as opposed to orthodox is that the property of *irreflexivity of the inequality relation* stimulated by the difference operator is not modally definable, but ‘ D_n definable’.

Now a carefully scrutiny reveals that D_n is precisely the *frame version* of the equations in [14, Theorem 3.2.5]. Using the notation in [9] (1):

$$D_n\phi = \bigvee_i \bigvee_{j \neq i} \Diamond_j(d_{ij} \wedge \Diamond_0 \dots \Diamond_{i-1} \Diamond_{i+1} \dots \Diamond_{n-1}\phi).$$

Here D_n is defined to make R^n its accessibility relation, that is for any n -square frame \mathbf{M} and $u \in \mathbf{M}$,

$$\mathbf{M}, u \models D_n\phi \iff (\exists v)(R^n(uv) \& v \models \phi).$$

The equations in the hypothesis of [14, Theorem 3.2.5] are (2):

$$\bigwedge_{k,l \in n} c_k[x \cdot y \cdot c_k(x \cdot -y)] - c_l(c_k x \cdot -d_{kl}) = 0.$$

(2) is the dual of (1) once one identifies the cylindrifier c_i the diamond modality \Diamond_i ($i < n$) together with some easy manipulations.

6.1 Rectangular density

Continue to fix $2 < n < \omega$. There is another algebraic expression of such non-orthodox derivation rules by so-called *density conditions*. This approach also has its roots in the monograph [14], cf. [14, Theorem 3.2.14] where it is proved that if $\mathfrak{A} \in \mathbf{CA}_n$ is an *atomic rectangularly dense algebra*, that is, \mathfrak{A} is atomic and its atoms are *rectangles*, then \mathfrak{A} is representable. Here a rectangle is an n -dimensional rectangle; it is an element that satisfies $\prod_{i < n} c_i x = x$. Notice that \geq always holds, because $x \leq c_i x$ for any $i < n$. When $n = 2$, and the algebra is a set algebra with top element $U \times U$, and $X \subseteq U \times U$ a 2-dimensional rectangle is just the familiar geometric rectangle obtained by cylindrifying on both dimensions. This is a nice sufficient condition for representability; in fact it can be proved that an algebra is representable \iff it can be embedded in an atomic one whose atoms are rectangles [14, Theorem 3.2.16]. The bad news is that this characterization cannot be expressed in a $\forall\exists$ formula due to the involvement of ‘atomicity’. From the

logical point of view this ‘complicated notion’ of atomicity is not warranted, because it does not lend itself to derivation rules even non-orthodox ones. The escape of this impasse was accomplished by simply *removing the condition* of atomicity by Andréka et al. [4], and proving that the representability result survives this omission. Every *rectangularly dense* algebra is representable, where by *rectangularly dense* is meant that below every element there is a rectangle, that is not necessarily an atom. Algebras considered might not be atomic. By the above discussion one cannot help but to ask the purely algebraic question. How are the notions of *rectangular density* and *richness*, related, if at all?

As illustrated next, both are algebraic expressions of a Henkin construction; the last achieving completeness for $L_{\omega,\omega}$. But instead they reflect algebraically a complete non-orthodox proof system for L_n using a variant of the difference operator.

The rectangles in a rectangularly dense algebra can be associated with so-called 0-thin elements [14, Definition 3.2.1]. Such elements have a double facet, a geometric one and a metalogical one. The metalogical interpretation is that these elements *abstract the notion of individual constants* [14, Remark 3.2.2]. Geometrically, in a set algebra 0-thin elements are obtained by fixing the first component of assignments by a constant. That is, if \mathfrak{A} is a set algebra with top element nU say, and $X \in \mathfrak{A}$ is 0 thin, then $X = \{s \in {}^\alpha U : s_0 = u\}$ for some $u \in U$. *Thinness here means that literally there is a thin line between the dimension of X and the dimension of \mathfrak{A} .* There is ‘enough supply’ of such elements in a rectangularly dense algebra. This algebraic notion of *richness* actually reflects the notion of *rich theories in Henkin constructions*.

Rich theories occurring in Henkin’s completeness proof eliminate existential quantifiers in existential formulas via individual constants more commonly referred to as *witnesses*. Algebraically, every cylindrifier is eliminated, or *witnessed* by a 0-thin element [14, Definition 3.2.1]. But then by algebraising the rest of Henkin’s proof, we get that rich algebras are representable. This connection manifests itself blatantly in the proof of [14, Theorem 3.2.5] where the base of the representation actually consists only of 0-thin elements, whereas the generic canonical models in Henkin constructions consist of individual constants. More succinctly, richness and rectangular density are *saturation conditions*. Geometrically: rectangular density means that below every non-zero element there is a rectangle, while richness means that below every element there is a *square*, a special kind of rectangle as the name suggests. The latter notion strikes one as weaker, but both notions are sufficiently strong to enforce representability, and both notions *can be transferred to non-orthodox derivation rules using the difference operator* achieving completeness when added to the cylindric (modal) axioms. In fact, it can (and was) proved in [4] that both notions are essentially equivalent. In *op.cit* it is proved that \mathfrak{A} is rectangularly dense (and quasi-atomic) $\iff \mathfrak{A}$ is rich.

For diagonal free cylindric algebras, Df_n for short, one cannot express the difference operator because there is no ‘equality’ in the signature, so that we *cannot express non-equality*. Rectangular density can be formulated in the language of Df_n but richness cannot be. So here one uses rectangular density to prove representability. These ideas were implemented by Venema [43] extending the results in [4] to Df_n s. The required representation (completeness theorem) was attained using *Rectangular games* played on so-called *crystal networks*. In the present ‘diagonal-free’ context the technique used by Andréka et al. [4] based on [14, Theorem 3.2.16] does not work.

Dually, from the (multi) modal logic perspective one takes the non-orthodox Gabbay–

like irreflexive (density) rule: $p \wedge \tau(\neg\phi \wedge p) \rightarrow \phi$, if $p \notin \phi$, then deduce ϕ , [9, p.1563], where

$$\tau(\chi) = \neg\Diamond_0 \dots \neg\Diamond_{n-1}[(\bigwedge_{i \in n} \Diamond_0 \dots \Diamond_{i-1} \Diamond_{i+1} \dots \exists \Diamond_{n-1} \chi) \wedge \neg\chi,$$

χ a formula. Together with the S5 axioms for each $i < n$ this gives a complete and sound (non-orthodox) proof system \vdash . This is a *translation of the algebraic property rectangular density*, in the sense that if T is a theory, then the Tarski–Lindenbaum quotient algebra \mathfrak{M}_T , where the quotient is taken with respect to \vdash is rectangularly dense, hence representable.

The representability result ‘*rectangular density in a $\mathbf{Df}_n \implies$ representability*’ was proved using games played on networks defined next:

Definition 6.1. Let $2 < n < \omega$ and $\mathfrak{A} \in \mathbf{Df}_n$.

- (1) An \mathfrak{A} *pre-network* is a pair $N = (N_1, N_2)$ where N_1 is a finite set of nodes, and $N_2 : N_1^n \rightarrow \mathfrak{A}$ is a total map. N is *atomic* if $\text{rng } N \subseteq \text{At } \mathfrak{A}$. We write N for any of N, N_1, N_2 relying on context, we write $\text{nodes}(N)$ for N_1 .
- (2) A pre-network N is said to be a network if it satisfies the following consistency condition: for $\bar{x}, \bar{y} \in N$ and $i < n$, $\bar{x} \equiv_i \bar{y} \implies N(\bar{x}) \cdot c_i N(\bar{y}) \neq 0$.

We define a game. But first a piece of notation. For a function f and $i \in \text{dom}(f)$, $g = f_i^u$ denotes the function with the same domain as f , such that $f \upharpoonright \text{dom}(f) \sim \{i\} = g \upharpoonright \text{dom}(f) \sim \{i\}$ and $g(i) = u$. A play of the game consists of a sequence $N_0 \subseteq N_1 \subseteq \dots$ of networks so that there are ω rounds. Suppose we are at round t , with the network N_t the outcome of the play so far. \forall makes a move by

- (1) Choosing $a \in \mathfrak{A}$. \exists must respond with a network $N_{t+1} \supseteq N_t$ such that either $N_{t+1}(\bar{x}) \leq a$ or $N_{t+1}(\bar{x}) \leq -a$,
- (2) \forall may choose an edge \bar{x} of N_t an index $i < n$ and $b \in \mathfrak{A}$ with $N_t(\bar{x}) \leq c_i b$. \exists must respond with a network $N_{t+1} \supseteq N_t$ such that for some $z \in N_{t+1}$, $N_{t+1}(\bar{x}_z^i) = b$.

Fix $2 < n < \omega$. Our aim is to show that if \mathfrak{A} is countable and rectangularly dense then \exists has a winning strategy and so \mathfrak{A} is representable. A rectangle in an $\mathfrak{A} \in \mathbf{Df}_n$, and a rectangularly dense \mathbf{Df}_n are defined like above (the definitions are free from diagonal elements).

Definition 6.2. A network N is said to be *rectangular* if $N(\bar{x})$ is a non-zero rectangle for every $s \in {}^n M$. M is a *crystal network* if $c_i M(s) = c_i M(t)$ whenever $s \equiv_i t$ for any $i < n$.

Lemma 6.3. [43, Lemmas 1, 2, p.1557, p.1560]. Let $\mathfrak{A} \in \mathbf{Df}_n$ be countable and rectangularly dense.

- (1) If M is a crystal network, and $r \in \mathfrak{A}$ is a non-zero rectangle below $M(t)$ for some $t \in {}^n M$, then there is a crystal extension M' of M such that $M'(t) = r$.
- (2) If M is crystal network and $c_i M(t) = c_i r$ for some $t \in {}^n M$ and some rectangle $r \in \mathfrak{A}$, $r > 0$, then there is a crystal extension M' of M and an element $t' \in {}^n M'$ such that $t \equiv_i t'$ and $M'(t) = r$.

Proof. For the first part define like on [9, p.1558], $M'(t) = M(t) \cdot c_{(\Delta(t,u))}r$, where $\Delta(t, u) = \{i < n : t_i \neq u_i\}$. Then $\text{nodes}(M') = \text{nodes}(N')$. For the second part: If there is already a tuple in nM such that $t \equiv_i u$ and $M(u) = c_i r$, then M is as required. Else, choose $k \notin M$ and let $K = M \cup \{k\}$ be defined as in [9, p.1560]. \square

Then it can be shown [43, Lemma 3], that in any round of the game, if the last network played is a crystal network then \exists can survive this round responding with a crystal network. From this we conclude using the arguments in [43] having at our disposal lemma 6.3:

Theorem 6.4. *Let $2 < n < \omega$. Let \mathfrak{A} be a countable rectangularly dense Df_n . Then \exists has a winning strategy in G_ω , hence $\mathfrak{A} \in \text{RDf}_n$.*

Proof. Like the arguments used above in constructing a representation as a limit of the play on networks cf. [43, Theorem 2, Lemma 3] using the crucial lemma 6.3. \square

The notions of density can be retrieved ‘modally’ using the difference operator, but they cannot be retrieved by finite Hilbert style axiomatizations. Such non-orthodox derivation rules can be traced back to the work of Gabbay and ever since have been frequently used by modal logicians, though some people argue that they capture extra variables or ‘witnesses’ from the back door and this is inimical to the modal nature. These extremely liberal Gabbay-style inference systems typically correspond to classes that are inductive, i.e., axiomatized by $\forall\exists$ -formulas. like the class of rectangularly dense cylindric algebras [4] and its diagonal free reducts.

We hasten to add that the ‘game technique’ used by Venema [43] works for all other cylindric-like algebras dealt with in [4] like Pinter’s substitution algebras and quasi-polyadic algebras with and without equality, but *not* ‘the other way round.’ The main technique in [4] which reduces the problem to the ‘atomic case’ already proved by Henkin et al. [14] (for cylindric algebras), does not settle the Df case.

But in any case the important thing here is that we have a sound and complete proof system, involving finite many axioms (and non-orthodox rules) achieving completeness relative to square semantics. Such a completeness theorem circumvents the dominating so resilient non-completeness theorems (obtained algebraically) by Monk, and considerably sharpened and refined by pioneers, to name a few, Andr  ka, Biro, Hirsch, Hodkinson and Maddux [2, 19, 26, 28, 33, 9]. Next we describe other efficient ways to avoid strong non-finite axiomatizability results.

7 Finitizability using guarding conjuncted with the semi-group approach

We start by a strong incompleteness theorem for the so-called *typeless logics* cf. [14, §4.3] indicating a ‘severe infinite’ mismatch between syntax and the intended classical semantics (in any attempt to give an algebraizable formalism of $L_{\omega,\omega}$), as long as we insist on full fledged commutativity of quantifiers. Here the adjective ‘typless’ indicates that in the formation rules of formulas, one drops the rank function specifying the arity of relation symbols; they all have the same rank, namely, ω . This is necessary if we want to algebraize $L_{\omega,\omega}$.

In a while we will manipulate *both syntax and semantics*. We will expand the syntax (vocabularly) with finitely many connectives, and reformulate it in such a way that the connectives are *finite* without *losing the expressive power of the previous formalism*; the *other (infinitely many) connectives (like all cylindrifiers $\exists_i : i < \omega$) are still there; they are definable from the finitely many new connectives*. We simultaneously *guard* (Tarskian square) semantics. In this way, we shall be able to obtain a strong completeness theorem for another algebraizable typeless variant of $L_{\omega,\omega}$ via a complete finite Hilbert style axiomatization involving only ‘type free’ valid formulas. In this process, we weaken commutativity of quantifiers and the intended semantics is accordingly ‘widened’ allowing *generalized models* where assignments are restricted to the *admissible ones*, in the sense of the coming definition 7.4. It is worthwhile sacrificing (but not fully) this ‘semantical’ precarious property of full-fledged commutativity of quantifiers seeing as how the harvest we reap is a strong completeness theorem for certain *tamed* finitary logics of infinitary relations having quite strong expressive power and, even more, unlike $L_{\omega,\omega}$ possibly having a decidable validity problem.

Let us start from the beginning, namely, by negative results. We start by defining the ‘classical’ more basic algebraizable typeless extension of $L_{\omega,\omega}$ with usual Tarskian square semantics in the context of an algebraizable logic (in the standard Blok–Pigozzi sense). Such logics are dealt with in [14, §4.3]. Our formulation is slightly more updated. By a *logical system*, a *logic for short* we understand a quadruple $(F, \mathbf{K}, \text{mng}, \models)$ where F is a set (of formulas) in a certain signature, \mathbf{K} is a class of structure mng is a function with domain $F \times \mathbf{K}$ and $\models \subseteq F \times F$. Intuitively, \mathbf{K} is the class of structures for the signature at hand $\text{mng}(\phi, M)$ is the interpretation of ϕ in M (possibly relativized), and \models is the pure semantical relation determined by \mathbf{K} . Such a definition is too broad; now we restrict a(n algebraizable) logic to be:

Definition 7.1. A logic $(F, \mathbf{K}, \text{mng}, \models)$ with formula algebra \mathfrak{F} of signature t is *algebraizable* if:

- (1) A set $\text{Cn}\mathfrak{L}$ the logical connectives fixed and each $c \in \text{Cn}\mathfrak{L}$ finite rank determining the signature t ,
- (2) There is set P called atoms such that \mathfrak{F} is the term algebra or absolutely free algebra over P with signature t ,
- (3) $\text{mng}_M = \langle \text{mng}(\phi, M) : \phi \in F \rangle$ is an endomorphism on \mathfrak{F} ,
- (4) There is a derived binary connective \leftrightarrow and a nullary connective \top that is compatible with the meaning functions, so that for all $\psi, \phi \in F$, we have $\text{mng}(\phi) = \text{mng}(\psi) \iff M \models \phi \leftrightarrow \psi$ and $M \models \phi \iff M \models \phi \leftrightarrow \top$,
- (5) For each endomorphism h of \mathfrak{F} , $M \in \mathbf{K}$, there is an $N \in \mathbf{K}$ such $\text{mng}_N = \text{mng}_M \circ h$, so that validity is preserved by endomorphisms.

Item (5) is what guarantees that instances of valid formulas remain valid for a homomorphism applied to a formula ϕ amounts to replacing the atomic formulas in ϕ by formula schemes. This is a crucial property for a logic to allow *algebraization*. We are ready to describe the algebraizable modification of $L_{\omega,\omega}$ (first order *with equality*.) To simplify matters, we consider the variables to have order type ω (the least infinite ordinal).

A *signature* is a triple $\Lambda = (\omega, R, \rho)$ such that R and ρ are functions with common domain a cardinal $\beta \leq \omega$, and $\rho(i) \leq \omega$ for all $i \in \beta$; R_i is a relation symbol of arity $\rho(i)$ (i.e the arity can be infinite.)

Fix a signature having ω many relation symbols each of arity ω . The atomic formulas are of the form $v_k = v_l$ or $R_l(v_0, \dots, v_i, \dots)_{i < \omega}$, $l \in \omega$. Variables can appear only in their natural order. Such atomic formulas are called *restricted* [13]. For first order logic with equality the notion of restricted formulas is only an apparent restriction because any (usual) first order formula is (semantically) equivalent to a restricted one.

For the above typless formalism using restricted formulas, one can dispense with the use of variables altogether, since they only appear in their natural order. No information is lost this way; one might as well write simply R instead of the atomic formula $R(x_0, x_1 \dots, x_i \dots)_{i < \omega}$, (R a relation symbol of arity ω). This formalism, like the case when we have only finitely many variables, readily lends itself to a (infinite)–dimensional propositional modal formalism, if we view the infinitely many existensial quantifiers as diamonds.

The formation rules of formulas are like (ordinary) first order logic; we use the logical connectives \wedge for conjunction and we denote negation by \neg . If ϕ, ψ are formulas, $i < \omega$, then $\phi \wedge \psi$, $\neg\phi$, and $\exists v_i \phi$, briefly $\exists_i \phi^1$ are formulas.

The set of (restricted) formulas is denoted by Fm_r . A structure is pair $\mathbf{M} = (M, \mathbf{R})$ where M is a non–empty set, $\mathbf{R} : \omega \rightarrow {}^\omega M$, and the interpretation of R_i in \mathbf{M} , in symbols $R_i(\bar{x})^{\mathbf{M}}$ is \mathbf{R}_i ($i < \omega$). The class of al structures is denoted by \mathbf{K} . For a sequence $s \in {}^\omega M$, $\mathbf{M} \in \mathbf{K}$, and a formula ϕ we write $M \models \phi[s]$ if s satisfies ϕ in \mathbf{M} ; this too is defined exactly like in first order logic. Having defined the basic semantical notions \models is defined the usual way (like first order logic). We denote $\{s \in {}^\omega M : \mathbf{M} \models \phi[s]\}$ by $\phi^{\mathbf{M}}$. For $\phi \in \mathfrak{Fm}_r$ and $\mathbf{M} \in \mathbf{K}$, $\text{mng}(\phi, \mathbf{M}) = \phi^{\mathbf{M}}$. Let $\mathfrak{L}_\omega = (Fm_r, \mathbf{K}, \text{mng}, \models)$. \mathfrak{L}_ω is referred to as a *finitary logic of infinitary relations*. For provability we use the basic proof system in [14, p. 157, §4.3] which is a natural extension of a complete calculus for $L_{\omega, \omega}$ expressed in terms of restricted formula. To formulate our next strong incompleteness result we need.

Definition 7.2. A *formula schema* in a logic \mathfrak{L} is a formula $\sigma(R_1, \dots, R_k)$ where R_1, \dots, R_k are relation symbol. An ω *instance* or even simply an instance of σ is a formula of the form $\sigma(\chi_1, \dots, \chi_k)$ where χ_1, \dots, χ_k are formulas and each R_i is replaced by χ_i .

Here *type–free valid formula schema* is a new notion of validity defined by Henkin et al. [14, Remark 4.3.65, Problem 4.16], [19, p. 487].

Theorem 7.3. (1) For any $k \geq 1$, there is no finite schemata of \mathfrak{L}_ω whose set Σ of instances satisfies

$$\Sigma \vdash_{\omega+k} \phi \iff \vdash_{\omega+k+1} \phi.$$

(2) For any $k \geq 1$, there is no finite schemata of \mathfrak{L}_ω whose set Σ of instances satisfies

$$\models \phi \implies \Sigma \vdash_{\omega+k} \phi.$$

Proof. Observe that if ϕ is a formula of \mathfrak{L} and $\tau(\phi)$ is its corresponding term in the language of CA_ω as defined in [14, Definition 4.3.55], then $\text{SNr}_\omega \text{CA}_{\omega+k} \models \tau(\phi) = \mathbf{1} \iff \vdash_{\omega+k} \phi$.

¹ \Diamond_i if one prefers the modal formalism

Thus the existence of such a schemata would imply finite axiomatizability by equations of $\mathbf{SNr}_\omega \mathbf{CA}_{\omega+k+1}$ over $\mathbf{SNr}_\omega \mathbf{CA}_{\omega+k}$ since every schema translates into an equation in the language of \mathbf{CA}_ω . We show that this cannot happen. We use (the notation and) idea in [22, Theorem 3.1]. In fact, we prove more than needed allowing the order type of variables to be any infinite ordinal α . We show that for any positive $k \geq 1$, the variety $\mathbf{SNr}_\alpha \mathbf{CA}_{\alpha+k+1}$ is not axiomatizable by a finite schema over $\mathbf{SNr}_\alpha \mathbf{CA}_{\alpha+k}$. This gives the required result.

We start by the finite dimensional case, then we lift the construction to the transfinite. Fix $2 < m < n < \omega$. Let $\mathfrak{C}(m, n, r)$ be the algebra $\mathfrak{C}a(\mathbf{H})$ where $\mathbf{H} = H_m^{n+1}(\mathfrak{A}(n, r), \omega)$, is the \mathbf{CA}_m atom structure consisting of all $n+1$ -wide m -dimensional wide ω hypernetworks [19, Definition 12.21] on $\mathfrak{A}(n, r)$ as defined in [19, Definition 15.2]. Then $\mathfrak{C}(m, n, r) \in \mathbf{CA}_m$. Then for any $r \in \omega$ and $3 \leq m \leq n < \omega$, $\mathfrak{C}(m, n, r) \in \mathbf{Nr}_m \mathbf{CA}_n$, $\mathfrak{C}(m, n, r) \notin \mathbf{SNr}_m \mathbf{CA}_{n+1}$ and $\Pi_{r/U} \mathfrak{C}(m, n, r) \in \mathbf{RCA}_m$, cf. [19, Corollaries 15.7, 5.10, Exercise 2, pp. 484, Remark 15.13]. Take

$$x_n = \{f \in H_m^{n+k+1}(\mathfrak{A}(n, r), \omega); m \leq j < n \rightarrow \exists i < m, f(i, j) = \text{Id}\}.$$

Then $x_n \in \mathfrak{C}(n, n+k, r)$ and $c_i x_n \cdot c_j x_n = x_n$ for distinct $i, j < m$. Furthermore (*), $I_n : \mathfrak{C}(m, m+k, r) \cong \mathfrak{Al}_{x_n} \mathfrak{Rd}_m \mathfrak{C}(n, n+k, r)$ via the map, defined for $S \subseteq H_m^{m+k+1}(\mathfrak{A}(m+k, r), \omega)$, by

$$\begin{aligned} I_n(S) &= \{f \in H_m^{n+k+1}(\mathfrak{A}(n, r), \omega) : f \upharpoonright^{\leq m+k+1} m \in S, \\ &\quad \forall j (m \leq j < n \rightarrow \exists i < m, f(i, j) = \text{Id})\}. \end{aligned}$$

We have proved the (known) result for finite ordinals > 2 .

To lift the result to the transfinite, we proceed like in [22], using the same lifting argument in *op.cit.* Let α be an infinite ordinal. Let $I = \{\Gamma : \Gamma \subseteq \alpha, |\Gamma| < \omega\}$. For each $\Gamma \in I$, let $M_\Gamma = \{\Delta \in I : \Gamma \subseteq \Delta\}$, and let F be an ultrafilter on I such that $\forall \Gamma \in I, M_\Gamma \in F$. For each $\Gamma \in I$, let ρ_Γ be an injective function from $|\Gamma|$ onto Γ . Let \mathfrak{C}_Γ^r be an algebra similar to \mathbf{CA}_α such that $\mathfrak{Rd}^{\rho_\Gamma} \mathfrak{C}_\Gamma^r = \mathfrak{C}(|\Gamma|, |\Gamma|+k, r)$ and let $\mathfrak{B}^r = \Pi_{\Gamma/F \in I} \mathfrak{C}_\Gamma^r$. Then we have $\mathfrak{B}^r \in \mathbf{Nr}_\alpha \mathbf{CA}_{\alpha+k}$ and $\mathfrak{B}^r \notin \mathbf{SNr}_\alpha \mathbf{CA}_{\alpha+k+1}$. These can be proved exactly like the proof of the first two items in [22, Theorem 3.1]. The second part uses that the element x_n is *m-rectangular* and *m-symmetric*, in the sense that for all $i \neq j \in m$, $c_i x_n \cdot c_j x_n = x_n$ and $s_i^j x_n \cdot s_j^i x_n = x_n$ (This last two conditions are not entirely independent [14]). This is crucial to guarantee that in the algebra obtained after relativizing to x_n , we do not lose commutativity of cylindrifiers. The relativized algebra stays inside \mathbf{CA}_n .

We know from the finite dimensional case that $\Pi_{r/U} \mathfrak{Rd}^{\rho_\Gamma} \mathfrak{C}_\Gamma^r = \Pi_{r/U} \mathfrak{C}(|\Gamma|, |\Gamma|+k, r) \subseteq \mathbf{Nr}_{|\Gamma|} \mathfrak{A}_\Gamma$, for some $\mathfrak{A}_\Gamma \in \mathbf{CA}_{|\Gamma|+\omega} = \mathbf{CA}_\omega$. Let $\lambda_\Gamma : \omega \rightarrow \alpha + \omega$ extend $\rho_\Gamma : |\Gamma| \rightarrow \Gamma (\subseteq \alpha)$ and satisfy $\lambda_\Gamma(|\Gamma| + i) = \alpha + i$ for $i < \omega$. Let \mathfrak{F}_Γ be a $\mathbf{CA}_{\alpha+\omega}$ type algebra such that $\mathfrak{Rd}^{\lambda_\Gamma} \mathfrak{F}_\Gamma = \mathfrak{A}_\Gamma$. Then $\Pi_{\Gamma/F} \mathfrak{F}_\Gamma \in \mathbf{CA}_{\alpha+\omega}$, and we have proceeding like in the proof of item 3 in [22, Theorem 3.1]: $\Pi_{r/U} \mathfrak{B}^r = \Pi_{r/U} \Pi_{\Gamma/F} \mathfrak{C}_\Gamma^r \cong \Pi_{\Gamma/F} \Pi_{r/U} \mathfrak{C}_\Gamma^r \subseteq \Pi_{\Gamma/F} \mathbf{Nr}_{|\Gamma|} \mathfrak{A}_\Gamma = \Pi_{\Gamma/F} \mathbf{Nr}_{|\Gamma|} \mathfrak{Rd}^{\lambda_\Gamma} \mathfrak{F}_\Gamma = \mathbf{Nr}_\alpha \Pi_{\Gamma/F} \mathfrak{F}_\Gamma$. But $\mathfrak{B} = \Pi_{r/U} \mathfrak{B}^r \in \mathbf{SNr}_\alpha \mathbf{CA}_{\alpha+\omega}$ because $\mathfrak{F} = \Pi_{\Gamma/F} \mathfrak{F}_\Gamma \in \mathbf{CA}_{\alpha+\omega}$ and $\mathfrak{B} \subseteq \mathbf{Nr}_\alpha \mathfrak{F}$, hence it is representable (here we use the neat embedding theorem). The rest follows using a standard L6s argument.

Item (2) follows also from the above proof using the same reasoning since the above proof also gives that \mathbf{RCA}_ω cannot be axiomatized by a finite schema over $\mathbf{SNr}_\omega \mathbf{CA}_{\omega+k}$ for any $k > 0$. \square

Henkin, Monk and Tarski formulated the so-called *finitizability problem* FP this way:

Devise an algebraic version of predicate logic in which the class of representable algebras forms a finitely based variety [2, 8, 11, 29, 12, 33, 9, 43, 19, 39, 14].

The FP in a nut shell, seeks a Stone-like representability result for algebras of relations having infinite rank, equivalently a strong completeness theorem for modifications of \mathfrak{L}_α by either changing the syntax and /or guarding semantics obtaining an optimal fit

The method of ‘guarding semantics’ goes back to the seminal 1985 paper of Andr  ka and Thompson [8] which proves possibly the first (historically) strong positive result on finite schema axiomatizations of algebras of α -ary relations, α an ordinal, intercepting the path of a series of negative non-finite axiomatizability results. For $\alpha < \omega$, this schema is a finite set of equations axiomatizing the resulting class of representable algebras with diagonalizable top elements. The term *guarding* though was introduced much later in the seminal paper of Andr  ka et al. [3], where Andr  ka, V. Benthem, and N  meti propose the guarded fragments of first order logic. The article [9] is an excellent account on this ‘fruitful contact between Crs’s and guarded logics’.

One way to understand the subtle technique of guarding is to look from above. One asks oneself what expressive resources will typically lead to undecidability and tries to avoid it by manipulating semantics. This view often also diffuses non-finite axiomatizability results. For the infinite dimensional case we succeed to obtain a strong finite axiomatizability result by guarding semantics, but whether this yields decidability of the validity problem with respect to this guarding is not settled in this paper.

To get rid of the ‘severe incompleteness’ result just proved for the common algebraic formalism of first order logic, one guards semantics in the infinite dimension case, too, in analogy to the finite dimensional case. More specifically, we will obtain an exact infinite analogue of the variety \mathbf{G}_n ; significantly distinct from \mathbf{G}_ω in that it admits a strictly finite axiomatization. We start from the logic side. The following theorem relates the semantics of a (possibly infinitary) formula ϕ in a *generalized model* to the semantics of its *guarded version*, denoted by $\text{guard}(\phi)$, in the standard part of the model expanded with the guard. Let $\alpha \leq \omega$. Let \mathfrak{L}_α denote the algebraizable formalism corresponding to \mathbf{CA}_α as defined above allowing a sequence of α -many variables, cf. [14, §4.3]. We dealt with the special \mathfrak{L}_ω in the statement of the previous theorem (but the proof covered \mathfrak{L}_α for any infinite ordinal α .) By induction on the complexity of formulas the following can be proved:

Theorem 7.4. *Let L be a signature taken in \mathfrak{L}_α . Let (M, V) be a generalized model in L , that is, M is an L -structure and $V \subseteq {}^\alpha M$ is the set of admissible assignments. Assume that R is an α -ary relation symbol outside L . For ϕ in L , let $\text{guard}(\phi)$ be the formula obtained from ϕ by relativizing all quantifiers to one and the same atomic formula $R(\bar{x})$ and let $\text{Guard}(M, V)$ be the model expanding M to $L \cup \{R\}$ by interpreting R via $R(s) \iff s \in V$. Then the following holds:*

$$M, V, s \models \phi \iff \text{Guard}(M, V), s \models \text{guard}(\phi),$$

where $s \in V$ and ϕ is a formula.

For α finite we have already dealt with the α -variable guarded fragments of $L_{\omega, \omega}$ where the admissible assignments were sets of the form $V \subseteq {}^\alpha U$, U a non-empty set. We considered the two cases when V is diagonalizable and when V is locally square. Such choices of ‘guards’ gave the finitely axiomatizable varieties \mathbf{D}_α and \mathbf{G}_α whose modal (set)

algebras have top elements V as specified, respectively. Such varieties have a decidable universal theory, too, witness theorem 5.1 and [8, 12, 39].

To get finite axiomatizability we use the so-called *semigroup approach* in algebraic logic that proved efficient in solving the FP for first order logic without equality. The choice of the semigroup controls the signature of our algebras expanding the signature of \mathbf{CA}_ω s. One takes only those substitution operators indexed by transformations in this fixed in advance subsemigroup, call it \mathbf{T} of $({}^\omega\omega, \circ)$ to be interpreted in set algebras the usual way like in polyadic set algebras. If \mathbf{T} happens to be finitely presented; for a start one gets a finite signature and a potential complete finite axiomatization. This finite signature *expands* that of \mathbf{CA}_ω in the sense that the infinitely many operations of \mathbf{CA}_ω become term definable in this (finite new) signature. But this potential can (and will) be attained. In this hitherto obtained finite signature, one stipulates a *finite* set of equations, enforcing representability (in the polyadic equality sense). This representability result is proved using a neat embedding theorem analogous to Henkin's neat embedding theorem for \mathbf{CA}_ω s; it is a relativized version thereof. In this context, cylindrifiers and diagonal elements are interpreted in the representing class of set algebras like in cylindric set algebras, and the substitutions operators, as indicated, are interpreted like in polyadic set algebras.

The difference here is that the top elements of the newly obtained set algebras are a union of ω -dimensional cartesian spaces that may not be disjoint. But such set algebras plainly share the intuitive geometric nature of generalized cylindric set algebras of the same dimension ω , whose top elements are unions of such ω -dimensional cartesian spaces that *are disjoint*. Dropping the condition of disjointness kills commutativity of cylindrifiers, but not completely. In this new guarded semantics a weaker commutativity property involving cylindrifiers and substitutions given in item \mathbf{C}_9^* of [12, Definition 6.3.7] is preserved.

We formulate the next theorem 7.5 fully proved in [39] as a Stone-like representability result for algebras of relations of infinite rank drawing the analogy with theorem 5.1. But first we give an example of a concrete *rich semigroup*; rather than giving the general definition. Such semigroups are defined (abstractly) in [29, 32]. The concrete instance recalled next makes the idea of *generating infinitely many operations using only finitely many* more tangible. The key idea here is the presence of a successor like element among the elements of the semigroup. Iterating this operator generates ω -many extra dimensions paving the way for a neat embedding theorem. (This idea can be traced back to the work of Craig addressing finitizability attempts as well).

The semigroup \mathbf{T} generated by the set of transformations $\{[i|j], [i, j], i, j \in \omega, \text{succ}, \text{pred}\}$ defined on ω is an example of a countable rich subsemigroup of $({}^\omega\omega, \circ)$. Here *succ* abbreviates the *successor function* on ω , $\text{succ}(n) = n+1$, and *pred* acts as its quasi-right inverse, the *predecessor function* on ω , defined by $\text{pred}(0) = 0$ and for other $n \in \omega$, $\text{pred}(n) = n - 1$. In every rich semigroup there are two elements π and σ called *distinguished elements* where π is a 'successor-like' transformation and σ is its quasi-inverse, of which the transformation *pred* is a special case.

Recall that $\mathcal{B}(V)$ denotes the Boolean algebra $\langle \wp(V), \cup, \cap, \sim, \emptyset, V \rangle$. The next theorem is an (algebraic) solution to the finitizability problem for first order logic with equality:

Theorem 7.5. *Let \mathbf{T} be a countable rich finitely presented subsemigroup of $({}^\omega\omega, \circ)$ with distinguished elements π and σ . Assume that \mathbf{T} is presented by the finite set of transformations S such that $\sigma \in S$. Then the class $\mathbf{Gp}_{\mathbf{T}}$ of all ω -dimensional set algebras of the*

form $\langle \mathcal{B}(V), C_0, D_{01}, S_\tau \rangle_{\tau \in S}$, where $V \subseteq {}^\omega U$, V a non-empty union of cartesian spaces, is a finitely axiomatizable variety. All the operations C_i, D_{ij} , $i, j \in \omega \sim \{0\}$ are term definable.

It is quite reasonable to take the concept of a *finite proof* as the most fundamental concept in logic. From this predominantly philosophically motivated point of view semantics come second and perhaps merely as a theoretical tool. Manipulating semantics to achieve finite Hilbert style axiomatizations no longer becomes mere tactical opportunism nor a challenging mental exercise, nor an intellectual enterprise. It rather becomes a pressing need.

In this respect, the logical counterpart of the first part of the previous theorem avoiding the severe incompleteness theorem obtained in theorem 7.3 is:

Theorem 7.6. *Let T be a semigroup as specified in the previous theorem. Let $\mathfrak{L}_{\mathsf{T}}$ be the algebraizable logic corresponding to Gp_{T} (in the Blok–Pigozzi sense). Then the satisfiability relation \models_w induced by Gp_{T} admits a finite recursive sound and complete proof calculus for the set of type-free valid formula schemata which involves only type-free valid formula schemata \vdash say, with respect to \models_w , so that $\Gamma \models_w \phi \iff \Gamma \vdash \phi$. This recursive complete axiomatization is a Hilbert style axiomatization, and there is a translation recursive function tr mapping $L_{\omega, \omega}$ formulas to formulas in $\mathfrak{L}_{\mathsf{T}}$ preserving \models_w (but not the usual validity \models).*

Here *type-free valid formula schemata* is the plural of *type-free valid formula schema*. This is a new notion of validity defined by Henkin et al. [14, Remark 4.3.65, Problem 4.16], [19, p. 487].

The solution in [39] where Tarskian semantics are broadened, but only slightly, possibly stands against Henkin, Monk and Tarski’s expectations, for the second problem in the first quote [13, pp.416] *does not* prohibit the option of changing the semantics, that is alter the notion of representability, as long as it is ‘concrete and intuitive’ enough. This, in turn, possibly indicates that their conjecture as formulated in the last two lines of their quote from [13, pp.416] (recalled above) was either too hasty or/ and unfounded. The aforementioned positive finitizability result formulated in theorem 7.5 is an infinite analogue of the polyadic equality analogue of the classical Andr  ka–Thompson–Resek theorem [8] proved by Ferenczi [12].

In set algebras considered the relativization (*guarding*) is the same.

For first order logic the *Entscheidungsproblem* posed by Hilbert has a negative answer: The validity problem of first order logic is undecidable. This is inherited by its finite variable fragments as long as the number of variables used is at least three. The validity problem for $\mathfrak{L}_{\mathsf{T}}$ is not known when T rich and finitely presented. There could be one such T that renders decidability. Algebraically, we do not know whether the equational theory of $\mathsf{IGp}_{\mathsf{T}}$ is decidable or not. Many examples of such semigroups are given in [29, 39].

We refer to the guarding dealt with in theorem 7.4 as *global guarding*. as opposed to *local guarding* a notion that we describe in the next.

Fix $2 < n < \omega$. We have dealt with the cases when $V = {}^n U$ (here there is no guarding) and the case when V is *locally square*; a guarding that we saw diffused the negative properties of ‘square Tarskian semantics’. This way of *global guarding* has led to the discovery of a whole landscape of multimodal logics having nice modal behaviour (like decidability) with the multimodal logic whose modal algebras are the class of relativized set algebras

‘at the bottom’ and Tarskian semantics with its undesirable properties (like undecidability) is only the top of an iceberg. Below the surface a treasure of nice multimodal logics was discovered. Viewed differently, first order logic with n variables is a *dynamic logic* of variable assignments, whose atomic processes shift values in registers x_0, \dots, x_{n-1} . This view opens up a hierarchy of fine structures underneath standard predicate logic using n variables, the latter becomes the undecidable theory of one particular mathematical class of ‘rich assignment models’ or squares. Furthermore, it lacks a completeness theorem using Hilbert style axiomatizations.

But there are other assignments models that are *not as rich, hence not as expressive* hence potentially decidable. Guarding can be viewed as the process of finding logics in this landscape that are reasonably expressive, share positive metalogical properties of first order, like interpolation and even improve on this by completeness and decidability. The idea is that we want to find a semantics that give just the right action while additional effects of square set-theoretic representations are separated out as negotiable decisions of formulation that can threaten completeness and decidability. Using square semantics is a voluntary commitment to one particular mathematical implication whose complexity seems to be an overkill. An insidious term often confuses this issue is the ‘concreteness of set theoretic models’ and the pre-assumption of the *canonicity* of ‘simple’ square ones. Such a conventional view can be harmful; preventing us from avoiding hereditary mistakes of old paradigms.

When we free ourselves from such prejudices, such ‘extended first order logics’ suggest further interesting applications. It becomes of practical interest just how much of predicate logic is used in mathematical proofs. Can decidable fragments of predicate logic be used instead? For example working logicians in linguistics or computer science have the gut feeling that the phenomena at hand are largely decidable. It is hard though to pin down mathematically such ‘feelings’ or intuition. Likewise there could be useful decidable systems of arithmetic or other parts of mathematics using such ideas. For example what is the theory of natural numbers with all possible families of variable assignments? The thrust of this line of research suggests that the *genuine logical core* of first order logic may well be decidable and that undecidability resulted from using ‘more than needed’. Proving decidability for guarded fragments of first order logic went historically via the *mosaic method* of Németi’s, later developed for guarded fragments to so-called quasi-models, which is a mixture of *filtration*, a well known technique for proving decidability results in modal logic, mosaics and semantic tableaux for first order logic. Such a technique also works for the loosely guarded fragments [9] the result we used in the proof of theorem 7.5.

7.1 Local guarding

Such locally guarded representations essentially amounts to working with the varieties $\mathbf{SNr}_n\mathbf{CA}_m$ ($n \leq m < \omega$) as defined in [13, Definitions 2.6.27] with m measuring how close or rather ‘how far’ we are from an ordinary representation. An algebra in $\mathbf{SNr}_n\mathbf{CA}_m$, for $n < m \leq \omega$ possesses a so-called *m-flat relativized representation* which is an ‘*m*-approximation’ to an ordinary representation. We stipulate that an ω -flat representation is just an ordinary one for algebras with countably many atoms. Roughly, the parameter m measures how much we have to ‘zoom in by a movable window’, so that we mistake the *m*-flat representation for an ordinary (genuine) one. Such notions are discussed and studied

extensively for relation algebras in [19, Chapter 13]. In [39] analogous investigations for cylindric-like algebras were initiated. To the best of our knowledge, such investigations (before [39]) seem to be lacking or at best very rare. By Henkin's Neat Embedding theorem, namely, using the notation of [13, Theorem 3.2.10], $\text{RCA}_n = \text{SNr}_n \text{CA}_\omega$ one might be tempted to attribute such negative results for $2 < n < \omega$, to the existence of infinitely many spare dimensions expressed in the superscript ω . But we shall see below that one should not run to such an unfounded and hasty conclusion. The presence of *only* $n + 3$ spare dimensions suffices to obtain negative results proved to hold for RCA_n . Indeed, as it happens, the classical negative results in [24, 18] on *atom-canonicity* and *complete representations* will be generalized to include the varieties $\text{SNr}_n \text{CA}_{n+k}$ for $2 < n < \omega$ and $k \geq 3$ in theorem 9.4.

Using the semantical notion of flatness, together with the results obtained in theorem 9.4 and those stated in the forthcoming theorem 7.8, we infer that for any $m \geq n + 3$ the variety of algebras having m -flat representations is not atom-canonical, and the class of algebras having *complete* m -flat representations is not first order definable. We learn from the above results, that for such proper approximations of RCA_n negative properties persist. Here by 'proper approximations' we mean that for $2 < n < \omega$ and finite $m > n$ $\text{RCA}_n \subsetneq \text{SNr}_n \text{CA}_m$ and $\bigcap_{k \in \omega, k \geq 3} \text{SNr}_n \text{CA}_{n+k} = \text{RCA}_n$ [13, Theorem 2.6.34]. Furthermore, for $2 < n < \omega$, the sequence $(\text{SNr}_n \text{CA}_m : n + 1 < m \leq \omega)$ is *strictly decreasing* (with respect to inclusion of classes) 'converging' to RCA_n , cf. [13, Problem 2.12] and its answer provided in [19, Theorem 15.1]. This last view is purely syntactical.

Till the end of this subsection fix $2 < n < m < \omega$.

Given $\mathfrak{A} \in \text{CA}_n$ having an m -flat representation with domain M , one forms an m -dilation \mathfrak{B} of \mathfrak{A} as follows: The algebra \mathfrak{B} has top element the so-called *n-Gaifmann hypergraph*, having set of hypereges the set $C^n(M) = \{s \in {}^m M : \text{rng}(s) \text{ is an } n\text{-clique}\}$; so \mathfrak{B} is a relativized set algebra of dimension m , with the operations in \mathfrak{B} are induced by the so-called *clique guarded semantics*; denoted by \models_c . Here an n -clique is a natural generalization of the notion of cliques in graph theory to hypergraphs; every n -tuple from $\text{rng}(s)$ is labelled by the top element of the algebra \mathfrak{A} .

Let $L(A)^n$ is the first order language using m variables in a signature consisting of one n -ary relation symbol for each element of \mathfrak{A} , cf. [19, Propsition 19.4–19.5].

Definition 7.7. The *clique guarded semantics* \models_c are defined inductively. For atomic formulas and Boolean connectives they are defined like the classical case and for existential quantifiers (cylindrifiers) they are defined as follows: for $\bar{s} \in {}^m M$, $i < m$, $M, \bar{s} \models_c \exists x_i \phi \iff$ there is a $\bar{t} \in C^n(M)$, $\bar{t} \equiv_i \bar{s}$ such that $M, \bar{t} \models \phi$. M is *m-square*, if $\bar{s} \in C^n(M)$, $a \in \mathfrak{A}$, $i < n$, and $l : n \rightarrow m$ is an injective map, $M \models c_i a(s_{l(0)}, \dots, s_{l(n-1)})$, \implies there is a $\bar{t} \in C^n(M)$ with $\bar{t} \equiv_i \bar{s}$, and $M \models a(t_{l(0)}, \dots, t_{l(n-1)})$.

Finally, M is said to be *m-flat* if it is m -square and for all $\phi \in \mathcal{L}(\mathfrak{A})^m$, for all $\bar{s} \in C^n(M)$, for all distinct $i, j < m$, $M \models_c [\exists x_i \exists x_j \phi \longleftrightarrow \exists x_j \exists x_i \phi](\bar{s})$.

So if M is an m -flat representation of \mathfrak{A} , then in the constructed m -dilation \mathfrak{B} of \mathfrak{A} with top element $C^n(M)$ and operations induced by the clique guarded (flat) semantics, cylindrifiers commute, so $\mathfrak{B} \in \text{CA}_m$, and hence $\mathfrak{A} \in \text{SNr}_n \text{CA}_m$.

Conversely, from an m -dilation \mathfrak{B} of the *canonical extension* of $\mathfrak{A} \in \text{CA}_n$, one constructs an m -dimensional hyperbasis [19, Definition 12.11] modified (in a straightforward manner) to the CA case. This m -dimensional hyperbasis can be viewed as a saturated set

of m -dimensional hypernetworks (mosaics) that can be glued together in a step-by-step manner to build the required m -flat representation of \mathfrak{A} . For the relation algebra case witness [19, Lemmata 13.33-34-35, Proposition 36].

We summarize the above discussion in the following theorem. For a class \mathbf{K} of having a Boolean reduct let $\mathbf{K} \cap \mathbf{At}$ be the class consisting of atomic algebras in \mathbf{K} . Now it is proved in [39] that:

Theorem 7.8. *Let $\mathfrak{A} \in \mathbf{CA}_n$. Then the following hold:*

1. $\mathfrak{A} \in \mathbf{SNr}_n \mathbf{CA}_m \iff \mathfrak{A}$ has an m -flat representation.
2. $\mathfrak{A} \in \mathbf{S}_c \mathbf{Nr}_n (\mathbf{CA}_m \cap \mathbf{At}) \iff \mathfrak{A}$ has a complete m -flat representation.

If an algebra \mathfrak{A} has an m -square representation, then the algebra neatly embeds into another m -dimensional algebra \mathfrak{B} . The ' m -dilation' \mathfrak{B} is formed the same way as for m -flatness. In particular, the top element of \mathfrak{B} is $C^n(M)$ where M is the m -square representation and the operations like before are induced by the clique guarded semantics. We have $\mathfrak{B} \in \mathbf{CrS}_m$, but \mathfrak{B} is not necessarily a \mathbf{CA}_m for it may fail commutativity of cylindrifiers. This discrepancy in the formed dilations blatantly manifests itself in a very important property. The 'Church Rosser condition' of commutativity of cylindrifiers in the formed dilation in case of m -flatness when $m \geq n + 3$, makes this clique guarded fragment strongly undecidable.

Let us formulate the latter result and some related ones for three dimensions. Assume that $m \geq 6$. Then it is undecidable to tell whether a finite algebra in \mathbf{CA}_3 has an m -flat representation, and the variety $\mathbf{SNr}_3 \mathbf{CA}_m$ cannot be finitely axiomatizable in k th order logic for any positive k . This can be proved by lifting the analogous results for relation algebras [19, Theorem 18.13, Corollaries 18.14, 18.15, 18.16]. One uses the construction of Hodkinson in [23] which associates recursively to every atomic relation algebra \mathbf{R} , an atomic $\mathfrak{A} \in \mathbf{CA}_3$ such that $\mathbf{R} \subseteq \mathbf{Ra}\mathfrak{A}$, the latter is the relation algebra reduct of \mathfrak{A} , cf. [14, Definition 5.3.7, Theorem 5.3.8]. The idea for the second part on non-finite axiomatizability is that the existence of any such finite axiomatization in k th order logic for any positive k , gives a decision procedure for telling whether a finite algebra is in $\mathbf{SNr}_3 \mathbf{CA}_m$ or not [19] which is impossible as just shown.

Furthermore, there are finite algebras that have infinite m -flat representations, but do not have finite ones, equivalently they *do not have a finite m -dimensional hyperbasis*. Roughly an m -dimensional hyperbasis consists of a 'saturated set' of m -dimensional hypernetworks. An m -dimensional hypernetwork is an extension of an m -dimensional networks with labelled hyperedges. These m -dimensional hypernetworks, in an m -dimensional hyperbasis, satisfy certain closure consistency conditions analogous to basic $m \times m$ matrices in an m -dimensional cylindric basis [21, Definition 12.11]. To see why, assume for contradiction that every finite algebra in $\mathbf{SNr}_3 \mathbf{CA}_m$ has a finite m -dimensional hyperbasis. We claim that there is an algorithm that decides membership in $\mathbf{SNr}_3 \mathbf{CA}_m$ for finite algebras which we know is impossible:

- Using a recursive axiomatization of $\mathbf{SNr}_3 \mathbf{CA}_m$ (exists), recursively enumerate all isomorphism types of finite \mathbf{CA}_3 s that are not in $\mathbf{SNr}_3 \mathbf{CA}_m$.
- Recursively enumerate all finite algebras in $\mathbf{SNr}_3 \mathbf{CA}_m$. For each such algebra, enumerate all finite sets of m -dimensional hypernetworks over \mathfrak{A} , using \mathbb{N} as hyperlabels,

and check to see if it is a hyperbasis. When a hyperbasis is located specify \mathfrak{A} . This recursively enumerates all and only the finite algebras in $\mathbf{SNr}_3\mathbf{CA}_m$. Since any finite \mathbf{CA}_3 is in exactly one of these enumerations, the process will decide whether or not it is in $\mathbf{SNr}_3\mathbf{CA}_m$ in a finite time.

We have shown that there are finite algebras that have infinite m -flat representations, but do not have finite ones (this cannot happen with m -squareness).

We end this section with the following theorem:

Theorem 7.9. *Let $\mathfrak{A} \in \mathbf{CA}_n$ and M be an m -flat representation of \mathfrak{A} . Then*

$$M, s \models_c \phi \iff M, s \models \text{packed}(\phi),$$

for all $s \in C^n(M)$ and every $\phi \in L(A)^n$, where $\text{packed}(\phi)$ denotes the translation of ϕ to the packed fragment [19, Definition 19.3].

In the sense of the previous theorem, the *clique guarded fragments*, which are the n -variable fragments of first order logic with clique (locally) guarded semantics are an alternative formulation of the *n -variable packed fragments* of first order logic [19, §19.2.3].

8 Products; a construction orthogonal to guarding

Now we present a construction that can be seen as *orthogonal to global guarding*. In the latter process *the states are altered* but the accessibility relations (corresponding to cylindrifiers and diagonal elements) *are the same*. In the present construction, the states are kept as they are but the *accessibility relations along the components are altered*.

Fix $2 < n < \omega$. In this subsection, though we follow [25] for terminology, the subsection is fairly self contained.

Combination of unimodal logics; fusions and products: Many multimodal logics can be considered as a combination of unimodal logics. So in a sense any result on multimodal logic sheds light on combining modal logics. But dually, we can start with ‘the components’ and form a modal logic that somehow encompasses them or extends them; we seek a multimodal logic in which they ‘embed’.

In such a process it is very natural to ask about *transfer results*, namely, these properties of the components that transfer to the combination, like axiomatizability (completeness), and decidability that involves complexity of the satisfiability problem.

There are two versions depending on whether the combination method is syntactic or semantical, namely, *fusion* and *products*, respectively. In fusions the components do not interact which makes transfer results from the components to the fusion easy to handle. In fact, the fusion of consistent modal logics is a conservative extension of the components, and the fusion of finitely many logics (this is well defined because the fusion operator is associative) has the finite model property if each of its components does. Fusion also preserves decidability. However, determining *degrees of complexity* is quite intricate here. For example, it is not known *PSPACE or EXPTIME completeness* transfer under formation of fusions.

Product logics are far more complex. They are designed semantically. A product logic is *the multimodal logic of products of Kripke complete frames*, so by definition it is

also Kripke complete. It is not hard to see that the fusion of logics is contained in their product. But in products *the modalities interact*, and this very interaction obviously adds to its components. In fusions such interactions are simply non-existent. In such a process negative properties persist. The reason basically is that products reflect the interaction of modalities; which is to be blamed for the negative result. If they miss on anything then they miss only on the uni-dimensional aspects of modalities and these do not really contribute to negative results. Negative results are caused by the interaction of modalities not by their uni-dimensional properties.

Transfer results: For example, in a product of two uni-modal logics, a precarious Church Rosser condition on modalities is created via $\Diamond_i \Box_j p = \Box_j \Diamond_i p$ ($i, j < n$). Indeed, it is known that the theory of two commuting confluence closure operators is undecidable. Nevertheless, *commuting closure operations* alone can be harmless like in the case of many cylindric-like algebras of dimension 2, but the interaction of the two modalities expressed by the confluence is potentially harmful. The product logic of two countable time flows is not even recursively enumerable, furthermore the modal logic of $(\mathbb{N}, <)$ is undecidable.

Compared to fusions, there are very few general transfer results for products, in fact here *the exact opposite occurs*. Nice properties do not transfer, rather the lack of transfer is the norm, particularly concerning finite axiomatizability and decidability. An interesting example here is \mathbf{K}^n for $n \geq 3$. Obviously the components are finitely axiomatizable, and their modal logics are decidable. However, such a logic viewed as a product of frames of the form (U, R) where $R \subseteq U \times U$ is an arbitrary relation has the finite model property, but it encodes the tiling problem and so it is undecidable.

In fact, for there are three dimensional formulas that are valid in all higher dimensional finite products, but can be falsified on an infinite frame. Here validity in higher dimensions is meaningful, because if $n < m$ then the modal logic \mathbf{K}^n embeds into \mathbf{K}^m . Furthermore, it is undecidable to tell whether a finite frame is a frame for this logic, and this gives strong non-finite axiomatizability results, and obviously implies undecidability. It is known [25] that between \mathbf{K}^n and $\mathbf{S5}^n$ for $n \geq 3$ is quite complicated. The class of modal algebras for $\mathbf{S5}^n$ is just the class of diagonal free cylindric algebras of dimension n . For an overview of such results, and more, the reader is referred to [25]. It is known [25] that one can add diagonal constants to n products forming so called δ products. The fact that all three dimensional modal logics are undecidable can be intuitively and indeed best explained by the undecidability of the product $\mathbf{S5}^3$ and its relation to the undecidable fragment of first order logic with 3 variables represented algebraically by \mathbf{CA}_3 whose equational theory is undecidable; a result of Maddux [14].

Square Tarskian semantics can be seen as a *limiting case of (i) relativized semantics, (ii) locally guarded or clique guarded semantics, (ii) products*

(i) In the first case the limit is taken on a varying set of worlds (states) approaching the square ${}^n U$.

(ii) Let $2 < n < m$. In the second case, viewed semantically m is allowed to grow. Here m -flatness witnesses of cylindrifiers are allowed more space. Syntactically more and more degrees of freedom or dimensions are created, so that an algebra $\mathfrak{A} \in \mathbf{CA}_n$ has an m -flat representation $\iff \mathfrak{A}$ neatly embeds into an m -dimensional algebra, which is a *truncated neat embedding theorem* as indicated above.

(iii) In products, *the accessibility relations along the components are changed* approaching $\mathbf{S5}^n$ where all accessibility relations along the components are the universal one.

Part 2, Getting more technical:

9 When rainbows and neat embeddings meet

Here we obtain results on classes of algebras having a neat embedding property using rainbow constructions. In the process, we generalize the seminal results on atom-canonicity and complete representations proved in [24, 18], respectively.

9.1 Rainbow constructions

We need the notions of *atomic networks* and *atomic games* [19, 21]:

Let $i < n$. For n -ary sequences \bar{x} and \bar{y} we write $\bar{x} \equiv_i \bar{y} \iff \bar{y}(j) = \bar{x}(j)$ for all $j \neq i$.

Definition 9.1. Fix finite $n > 1$.

- (1) An n -dimensional *atomic network* on an atomic algebra $\mathfrak{A} \in \mathbf{CA}_n$ is a map $N : {}^n\Delta \rightarrow \text{At}\mathfrak{A}$, where Δ is a non-empty set of *nodes*, denoted by $\text{nodes}(N)$, satisfying the following consistency conditions:

- If $\bar{x} \in {}^n\text{nodes}(N)$, and $i < j < n$, then $N(\bar{x}) \leq d_{ij} \iff x_i = x_j$.
- If $\bar{x}, \bar{y} \in {}^n\text{nodes}(N)$, $i < n$ and $\bar{x} \equiv_i \bar{y}$, then $N(\bar{x}) \leq c_i N(\bar{y})$.

For n -dimensional atomic networks M and N , we write $M \equiv_i N \iff M(\bar{y}) = N(\bar{y})$ for all $\bar{y} \in {}^n(n \sim \{i\})$.

- (2) Assume that $\mathfrak{A} \in \mathbf{CA}_n$ is atomic and that $m, k \leq \omega$. The *atomic game* $G_k^m(\text{At}\mathfrak{A})$, or simply G_k^m , is the game played on atomic networks of \mathfrak{A} using m nodes and having k rounds [21, Definition 3.3.2], where \forall is offered only one move, namely, a *cylindrifier move*:

- Suppose that we are at round $t > 0$. Then \forall picks a previously played network N_t ($\text{nodes}(N_t) \subseteq m$), $i < n$, $a \in \text{At}\mathfrak{A}$, $x \in {}^n\text{nodes}(N_t)$, such that $N_t(x) \leq c_i a$. For her response, \exists has to deliver a network M such that $\text{nodes}(M) \subseteq m$, $M \equiv_i N$, and there is $y \in {}^n\text{nodes}(M)$ that satisfies $y \equiv_i x$, and $M(y) = a$.

- (3) We write $G_k(\text{At}\mathfrak{A})$, or simply G_k , for $G_k^m(\text{At}\mathfrak{A})$ if $m \geq \omega$. The *atomic game* $F^m(\text{At}\mathfrak{A})$, or simply F^m , is like $G_\omega^m(\text{At}\mathfrak{A})$ except that \forall has the advantage to reuse the available n nodes during the play.

One can show that for $2 < n < m < \omega$, the game F^m tests neat embeddability in the following sense:

Lemma 9.2. [39] *If \mathfrak{A} is atomic and $\mathfrak{A} \in \mathbf{S_cNr}_n\mathbf{CA}_m$, then \exists has a winning strategy in $F^m(\text{At}\mathfrak{A})$. In particular, if $\mathfrak{A} \in \mathbf{Nr}_n\mathbf{CA}_\omega$, then \exists has a winning strategy in $F^\omega(\text{At}\mathfrak{A})$ and $G_\omega(\text{At}\mathfrak{A})$, and if \mathfrak{A} is finite and \forall has a winning strategy in $F^m(\text{At}\mathfrak{A})$, then $\mathfrak{A} \notin \mathbf{SNr}_n\mathbf{CA}_m$.*

Proof. [43, 39]. The last part follows by observing that for any $\mathfrak{C} \in \mathbf{CA}_n$, if $\mathfrak{C} \in \mathbf{SNr}_n\mathbf{CA}_m \implies \mathfrak{C}^+ \in \mathbf{S_cNr}_n\mathbf{CA}_m$ (where \mathfrak{C}^+ is the canonical extension of \mathfrak{C}) and if \mathfrak{C} is finite, then of course $\mathfrak{C} = \mathfrak{C}^+$. \square

In the first item of theorem 9.4, we will use the following lemma:

Lemma 9.3. *Let $2 < n < m < \omega$. Let $\mathfrak{A} \in \mathbf{CA}_n$ be finite. Then \exists has a winning strategy in $G^m(\text{At}\mathfrak{A}) \iff \mathfrak{A}$ has an m -square representation.*

Now we recall ‘rainbow constructions’ as introduced in algebraic logic by Hirsch and Hodkinson [21]. Fix $2 < n < \omega$. *Coloured graphs* [18] are complete graphs whose edges are labelled by the rainbow colours, \mathbf{g} (greens), \mathbf{r} (reds), and \mathbf{w} (whites) satisfying certain consistency conditions. The greens are $\{\mathbf{g}_i : 1 \leq i < n-1\} \cup \{\mathbf{g}_0^i : i \in \mathbf{G}\}$ and the reds are $\{\mathbf{r}_{ij} : i, j \in \mathbf{R}\}$ where \mathbf{G} and \mathbf{R} are two relational structures. The whites are $\mathbf{w}_i : i \leq n-2$.

In coloured graphs certain triangles are forbidden. For example a green triangle (a triangle whose edges are all green) is forbidden. Not all red triangles are allowed. In consistent (allowed) red triangle the indices ‘must match’ satisfying a certain ‘consistency condition’. Also, in coloured graphs some $n-1$ tuples (hyperedges) are labelled by shades of yellow [18].

Given relational structures \mathbf{G} and \mathbf{R} the rainbow atom structure of dimension n are equivalence classes of surjective maps $a : n \rightarrow \Delta$, where Δ is a coloured graph in the rainbow signature, and the equivalence relation relates two such maps \iff they essentially define the same graph [18, 4.3.4]; the nodes are possibly different but the graph structure is the same. We let $[a]$ denote the equivalence class containing a . The accessibility binary relation corresponding to the i th cylindrifier ($i < n$) is defined by: $[a]T_i[b] \iff a \upharpoonright n \sim \{i\} = b \upharpoonright n \sim \{i\}$, and the accessibility unary relation corresponding to the ij th diagonal element ($i < j < n$) is defined by: $[a] \in D_{ij} \iff a(i) = a(j)$. We refer to the atom $[a]$ ($a : n \rightarrow \Delta$) as an atom and sometimes as an n -coloured graph. We denote the complex algebra of the rainbow atom structure based on \mathbf{G} and \mathbf{R} by $\mathbf{CA}_{\mathbf{G},\mathbf{R}}$. The dimension of $\mathbf{CA}_{\mathbf{G},\mathbf{R}}$ will be clear from context. Identifying $[a]$ with a , we sometimes refer to an atom $[a] : n \rightarrow \Delta$ of $\mathbf{CA}_{\mathbf{G},\mathbf{R}}$ as an n -coloured graph.

If \mathfrak{A} is an (atomic) rainbow \mathbf{CA}_n , and $k, m \leq \omega$, then the games $G_k(\text{At}\mathfrak{A})$ and $F^m(\text{At}\mathfrak{A})$ translate to games on coloured graphs (CGR) [18, p.27–29]. Certain special finite coloured graphs play an essential role in ‘rainbow games’ whose board consists of coloured graphs. Such special coloured graphs are called *cones*:

*Let $i \in \mathbf{G}$, and let M be a coloured graph consisting of n nodes x_0, \dots, x_{n-2}, z . We call M an i -cone if $M(x_0, z) = \mathbf{g}_0^i$ and for every $1 \leq j \leq n-2$, $M(x_j, z) = \mathbf{g}_j$, and no other edge of M is coloured green. (x_0, \dots, x_{n-2}) is called the **base of the cone**, z the **apex of the cone** and i the **tint of the cone**.*

The winning strategy of \forall in the rainbow game played on coloured graphs played between \exists and \forall is bombarding \exists with i -cones, $i \in \mathbf{G}$, having the same base and distinct green tints. To respect the rules of the game \exists has to choose a red label for appexes of two successive cones. Eventually, running out of ‘suitable reds’, \exists is forced to play an inconsistent triple of reds where indices do not match. Thus \forall wins on a red clique (a graph all of whose edges are labelled by a red). Such a winning strategy is dictated by a simple Ehrenfeucht–Fraïssé game played on the relational structures \mathbf{G} and \mathbf{R} denoted by $\text{EF}_p^r(\mathbf{G}, \mathbf{R})$ in [21, Definition 16.2] where r is the number of rounds and p is number of pairs of pebbles on board.

9.2 Atom–canonicity, first order definability and finite axiomatizability

Unless otherwise indicated, in this section n is fixed to be finite $n > 2$. To formulate and prove the next theorem, we need to fix some notation. We write \mathbf{S}_c for the operation of forming complete subalgebras and \mathbf{S}_d for the operation of forming dense subalgebras. Let \mathbf{K} be a class of BAOs and $\mathfrak{A}, \mathfrak{B} \in \mathbf{K}$ such that $\mathfrak{A} \subseteq \mathfrak{B}$. Then \mathfrak{A} is a *complete subalgebra* of \mathfrak{B} if $\mathfrak{A} \subseteq \mathfrak{B}$ and for $X \subseteq \mathfrak{A}$, $\sum^{\mathfrak{A}} X = 1 \implies \sum^{\mathfrak{B}} X = 1$. (Recall that) \mathfrak{A} is *dense* in \mathfrak{B} if $(\forall b \in B \sim \{0\})(\exists a \in \mathfrak{A} \sim \{0\})(a \leq b)$. Observe that the two definitions depend only on the Boolean part of the algebras considered. Furthermore, it is not hard to show that for any such \mathbf{K} , $\mathbf{K} \subseteq \mathbf{S}_d \mathbf{K} \subseteq \mathbf{S}_c \mathbf{K}$. In fact, the last two inclusions are proper when \mathbf{K} is the class of Boolean algebras (without operators).

Now we are ready to generalize the main results in [24, 18]. In the last reference it is shown that the class of completely representable \mathbf{CA}_n s, briefly \mathbf{CRCA}_n when $2 < n < \omega$ is not elementary, and in the former, it is shown that \mathbf{RCA}_n is not atom–canonical. Recall that Andr eka [2] splits atoms in a representable algebra to get a non–representable one. But one can split atoms to do the exact opposite. One can undergo *the reverse process* passing from a non–representable algebra to a representable one via so–called *blow up and blur constructions* [7, 39], a task to be implemented next. Recall that n is fixed to be finite > 2 . Let $k \geq 3$. For an infinite ordinal α , we stipulate that a (complete) α –flat representation is just a (complete) representation.

Theorem 9.4. 1. *The variety $\mathbf{SNr}_n \mathbf{CA}_{n+k}$ is not atom–canonical. Hence the variety of algebras having $n+k$ –flat representations is not atom–canonical. In particular, \mathbf{RCA}_n is not atom–canonical [24]. Furthermore, the variety of algebras having $n+k$ –square representations is not atom–canonical.*

2. *For any class \mathbf{K} , such that $\mathbf{CRCA}_n \cap \mathbf{S}_d \mathbf{Nr}_n \mathbf{CA}_\omega \subseteq \mathbf{K} \subseteq \mathbf{S}_c \mathbf{Nr}_n \mathbf{CA}_{n+k}$, \mathbf{K} is not elementary. The class of algebras having complete $n+k$ –flat representations is not elementary. Furthermore, any class \mathbf{L} such that $\mathbf{At}(\mathbf{Nr}_n \mathbf{CA}_\omega) \subseteq \mathbf{L} \subseteq \mathbf{At}(\mathbf{S}_c \mathbf{Nr}_n \mathbf{CA}_{n+3})$ is not elementary. Finally, $\mathbf{ElNr}_n \mathbf{CA}_\omega \not\subseteq \mathbf{S}_d \mathbf{Nr}_n \mathbf{CA}_\omega \iff$ any class \mathbf{L} such that $\mathbf{Nr}_n \mathbf{CA}_\omega \subseteq \mathbf{L} \subseteq \mathbf{S}_c \mathbf{Nr}_n \mathbf{CA}_{n+3}$, \mathbf{L} is not elementary.*

Proof. First item: One starts with a finite algebra $\mathfrak{C} \in \mathbf{CA}_n$ that is outside $\mathbf{SNr}_n \mathbf{CA}_{n+3}$. This algebra is the rainbow algebra $\mathbf{CA}_{n+1,n}$. To see why it is outside $\mathbf{SNr}_n \mathbf{CA}_{n+3}$, consider the game Ehrenfeucht–Fra iss e forth game $\mathbf{EF}_r^p(n+1, n)$. In this game played on the complete irreflexive graphs $n+1$ and n , \forall has a winning strategy since $n+1$ is ‘longer’. Here r is the number of rounds and p is the number of pairs of pebbles on board. Using (any) $p > n$ many pairs of pebbles available on the board \forall can win this game in $n+1$ many rounds. In each round $0, 1 \dots n$, \exists places a new pebble on a new element of $n+1$. The edge relation in n is irreflexive so to avoid losing \exists must respond by placing the other pebble of the pair on an unused element of n . After n rounds there will be no such element, so she loses in the next round. It is not hard to show that the winning strategy of \forall in the private Ehrenfeucht–Fra iss e game lifts to a winning strategy in the graph game $G_k^{n+3}(\mathbf{At}(\mathbf{CA}_{n+1,n}))$ [18, pp.841] for some finite k . Hence by lemma 9.3, $\mathbf{CA}_{n,n+1}$ has no $n+3$ –square representation. But plainly \forall has a winning strategy in $F_k^{n+3}(\mathbf{At}(\mathbf{CA}_{n+1,n}))$ (same k) from which it follows using lemma 9.2 that $\mathbf{CA}_{n+1,n} \notin \mathbf{SNr}_n \mathbf{CA}_{n+3}$.

Then one *blows up and blur* $\mathbf{CA}_{n+1,n}$, by splitting its ‘(red) atoms’ each to infinitely many. After this splitting one gets a new infinite atom structure \mathbf{At} . This atom structure

is like the atom structure of the (set) algebra \mathfrak{A} constructed in [24, Definition 4.1, §4.2] except that the number of greens having 0 as a subscript involved (in constructing \mathbf{At}) is only $n + 1$, rather than ω -many, as is the case in the construction of [24].

So in the present context, after the splitting ‘the finitely many red colours’ replacing each such red colour r_{kl} , $k < l < n$ by ω many r_{kl}^i , $i \in \omega$, the rainbow signature for the resulting rainbow theory as defined in [19, Definition 3.6.9] call this theory T_{ra} , consists of $\mathbf{g}_i : 1 \leq i < n - 1$, $\mathbf{g}_0^i : 1 \leq i \leq n + 1$, $\mathbf{w}_i : i < n - 1$, $r_{kl}^t : k < l < n$, $t \in \omega$, binary relations, and $n - 1$ ary relations \mathbf{y}_S , $S \subseteq_\omega n + k - 2$ or $S = n + 1$.

The representation of \mathfrak{ImAt} can be built exactly like in [24]. The representing set algebra of dimension n has base an n -homogeneous model M of another theory T whose signature expands that of T_{ra} by an additional binary relation (a shade of red) ρ . In this new signature T is obtained from T_{ra} by some axioms (consistency conditions) extending T_{ra} . Such axioms (consistency conditions) specify consistent triples involving ρ . This model M is constructed as a countable limit of finite models of T using a game played between \exists and \forall . We call the models of T extended coloured graphs. In particular, M is an extended coloured graph. Here, unlike the extended $L_{\omega_1, \omega}$ theory dealt with in [24], T is a *first order one* because the number of greens used are finite.

As is the case with ‘rainbow games’ [18, 19] \forall challenges \exists with *cones* having green *tints* (\mathbf{g}_0^i), and \exists wins if she can respond to such moves. This is the only way that \forall can force a win. \exists has to respond by labelling *appexes* of two successive cones, having the *same base* played by \forall . By the rules of the game, she has to use a red label. The winning strategy is implemented by \exists using the red label ρ (outside the rainbow signature) that comes to her rescue whenever she runs out of ‘rainbow reds’, so she can respond with an extended coloured graphs. It turns out inevitable, that some edges in M are labelled by ρ during the play; in fact these edges labelled by ρ will form an *infinite red clique* (an infinite complete extended graph whose edges are all labelled by ρ .)

Now $\mathbf{CA}_{n, n+1}$ embeds into \mathfrak{ImAt} by mapping every n -coloured graph to the join of its copies. Let us describe the embedding more rigorously. Let \mathbf{CRG}_f denote the class of coloured graphs on $\mathbf{AtCA}_{n, n+1}$ and \mathbf{CRG} be the class of coloured graph on $\mathbf{At} = \mathbf{At}\mathfrak{A}$. For the sake of brevity denote $\mathbf{AtCA}_{n, n+1}$ by \mathbf{At}_f . We can assume that $\mathbf{CRG}_f \subseteq \mathbf{CRG}$. Write M_a for the atom that is the (equivalence class of the) surjection $a : n \rightarrow M$, $M \in \mathbf{CGR}$. Here we identify a with $[a]$; no harm will ensue. We define the (equivalence) relation \sim on \mathbf{At} by $M_b \sim N_a$, $(M, N \in \mathbf{CGR})$:

- $a(i) = a(j) \iff b(i) = b(j)$,
- $M_a(a(i), a(j)) = r^l \iff N_b(b(i), b(j)) = r^k$, for some $l, k \in \omega$,
- $M_a(a(i), a(j)) = N_b(b(i), b(j))$, if they are not red,
- $M_a(a(k_0), \dots, a(k_{n-2})) = N_b(b(k_0), \dots, b(k_{n-2}))$, whenever defined.

We say that M_a is a *copy* of N_b if $M_a \sim N_b$. We say that M_a is a *red atom* if it has at least one edge labelled by a red rainbow colour r_{ij}^l for some $i < j < n$ and $l \in \omega$. Clearly every red atom M_a has infinitely countable many red copies, which we denote by $\{M_a^{(j)} : j \in \omega\}$. Now we define a map $\Theta : \mathbf{CA}_{n+, n} = \mathfrak{ImAt}_f$ to \mathfrak{ImAt} , by specifying first its values on \mathbf{At}_f , via $M_a \mapsto \sum_j M_a^{(j)}$; each atom maps to the suprema of its copies.

If M_a is not red, then by $\sum_j M_a^{(j)}$, we understand M_a . This map is extended to $\mathbf{CA}_{n+k-2,n}$ by $\Theta(x) = \bigcup \{\Theta(y) : y \in \mathbf{AtCA}_{n+k-2,n}, y \leq x\}$. The map Θ is well-defined, because \mathfrak{CmAt} is complete. It is not hard to show that the map Θ is an injective homomorphism. (Injectivity follows from the fact that $M_a \leq f(M_a)$, hence $\Theta(x) \neq 0$ for every atom $x \in \mathbf{At}(\mathbf{CA}_{n+1,n})$.) These precarious joins prohibiting membership in $\mathbf{SNr}_n \mathbf{CA}_{n+3}$ *do not* exist in the term algebra \mathfrak{TmAt} , the subalgebra of \mathfrak{CmAt} generated by the atoms, only joins of finite or cofinite subsets of the atoms do.

We have shown that $\mathfrak{TmAt} \in \mathbf{RCA}_n$ but its Dedekind-MacNeille completion \mathfrak{CmAt} is outside $\mathbf{SNr}_n \mathbf{CA}_{n+3}$ and \mathfrak{CmAt} does not have an $n+3$ -square representation proving the required.

We give another approach to proving the above result on non atom-canonicity. Start by the rainbow atom structure \mathbf{At} (of the atomic set algebra \mathfrak{A}), whose atoms are surjections from n to coloured graphs (briefly n -coloured graphs). labelled by the rainbow colours (corresponding to the rainbow signature), greens, rainbow reds, whites..etc specified above. Weak representability of \mathbf{At} can be proved without the need of the additional red label ρ . The term algebra \mathfrak{TmAt} *can be represented intrinsically*, using the set of ultrafilters $\omega \cup \{\mathbf{Ruf}\}$ as ‘colours’, where ω ‘codes’ the countably many *principal ultrafilters* of \mathfrak{TmAt} ; each such ultrafilter is generated by an atom, namely, an n -coloured graph, and \mathbf{Ruf} is the ‘Red Ultrafilter’ of \mathfrak{TmAt} containing the filter generated by the co-finite sets of n -coloured red graphs.

Here \mathbf{Ruf} and ρ are *literally* two sides of the same coin. In fact, ρ (and \mathbf{Ruf}) will have to be used infinitely many times during the play so that the base of any *potential representation* of \mathfrak{CmAt} will have an infinite red clique $\mathbf{RC} = (a_n : n \in \omega)$, say, whose edges are labelled by ρ . It is precisely, this \mathbf{RC} that prohibited a representation of $\mathfrak{C} = \mathfrak{CmAt}$ as illustrated in some detail on [24, pp. 9–10]. though \mathbf{RC} exists in the representation of \mathfrak{TmAt} , since \mathfrak{TmAt} is not complete, \mathbf{RC} will not cause any trouble as far as the representability of \mathfrak{TmAt} is concerned.

(2) For the second item. Fix finite $n > 2$. One takes a rainbow-like algebra based on the ordered structure \mathbb{Z} and \mathbb{N} , that is similar but not identical to $\mathbf{CA}_{\mathbb{Z},\mathbb{N}}$; call this (complex) algebra \mathfrak{C} . The reds \mathbf{R} is the set $\{r_{ij} : i < j < \omega (= \mathbb{N})\}$ and the green colours used constitute the set $\{g_i : 1 \leq i < n-1\} \cup \{g_0^i : i \in \mathbb{Z}\}$. In complete coloured graphs the forbidden triples are like in usual rainbow constructions; more specifically the following are forbidden triangles in coloured graphs [18, 4.3.3]:

$$(g, g', g^*), (g_i, g_i, w_i), \quad \text{any } 1 \leq i \leq n-2 \quad (1)$$

$$(g_0^j, g_0^k, w_0) \quad \text{any } j, k \in \mathbb{G} \quad (2)$$

$$(r_{ij}, r_{j'k'}, r_{i^*k^*}) \quad \text{unless } i = i^*, j = j' \text{ and } k' = k^*, \quad (3)$$

but now the triple (g_0^i, g_0^j, r_{kl}) is also forbidden if $\{(i, k), (j, l)\}$ is not an order preserving partial function from $\mathbb{Z} \rightarrow \mathbb{N}$. It can be proved that \exists has a winning strategy ρ_k in the k -rounded game $G_k(\mathbf{AtC})$ for all $k \in \omega$ [39]. Hence, using ultrapowers and an elementary chain argument [21, Corollary 3.3.5], one gets a countable (completely representable) algebra \mathfrak{B} such that $\mathfrak{B} \equiv \mathfrak{A}$, and \exists has a winning strategy in $G_\omega(\mathbf{AtB})$.

On the other hand, one can show that \forall has a winning strategy in $F^{n+3}(\mathbf{AtC})$. The idea here, is that, as is the case with winning strategy's of \forall in rainbow constructions, \forall bombards \exists with cones having distinct green tints demanding a red label from \exists to

apexes of successive cones. The number of nodes are limited but \forall has the option to re-use them, so this process will not end after finitely many rounds. The added order preserving condition relating two greens and a red, forces \exists to choose red labels, one of whose indices form a decreasing sequence in \mathbb{N} . In ω many rounds \forall forces a win, so by lemma 9.2 $\mathfrak{C} \notin \mathbf{S}_c \mathbf{Nr}_n \mathbf{CA}_{n+3}$.

He plays as follows [39]: In the initial round \forall plays a graph M with nodes $0, 1, \dots, n-1$ such that $M(i, j) = w_0$ for $i < j < n-1$ and $M(i, n-1) = g_i$ ($i = 1, \dots, n-2$), $M(0, n-1) = g_0^0$ and $M(0, 1, \dots, n-2) = y_{\mathbb{Z}}$. This is a 0 cone. In the following move \forall chooses the base of the cone $(0, \dots, n-2)$ and demands a node n with $M_2(i, n) = g_i$ ($i = 1, \dots, n-2$), and $M_2(0, n) = g_0^{-1}$. \exists must choose a label for the edge $(n+1, n)$ of M_2 . It must be a red atom r_{mk} , $m, k \in \mathbb{N}$. Since $-1 < 0$, then by the ‘order preserving’ condition we have $m < k$. In the next move \forall plays the face $(0, \dots, n-2)$ and demands a node $n+1$, with $M_3(i, n) = g_i$ ($i = 1, \dots, n-2$), such that $M_3(0, n+2) = g_0^{-2}$. Then $M_3(n+1, n)$ and $M_3(n+1, n-1)$ both being red, the indices must match. $M_3(n+1, n) = r_{lk}$ and $M_3(n+1, n-1) = r_{km}$ with $l < m \in \mathbb{N}$. In the next round \forall plays $(0, 1, \dots, n-2)$ and re-uses the node 2 such that $M_4(0, 2) = g_0^{-3}$. This time we have $M_4(n, n-1) = r_{jl}$ for some $j < l < m \in \mathbb{N}$. Continuing in this manner leads to a decreasing sequence in \mathbb{N} .

But we can go further. We define another k -rounded atomic game stronger than G_k call it H_k , for $k \leq \omega$, so that if $\mathfrak{D} \in \mathbf{CA}_n$ is countable and atomic and \exists has a winning strategy in $H_\omega(\text{At}\mathfrak{D})$, then (*) $\text{At}\mathfrak{D} \in \mathbf{AtNr}_n \mathbf{CA}_\omega$ and $\mathfrak{CmAt}\mathfrak{D} \in \mathbf{Nr}_n \mathbf{CA}_\omega$ (these two conditions taken together do not imply that $\mathfrak{D} \in \mathbf{Nr}_n \mathbf{CA}_\omega$, witness example 9.6).

It can be shown that \exists has a winning strategy in $H_k(\text{At}\mathfrak{C})$ for all $k \in \omega$, hence using ultrapowers and an elementary chain argument, we get that $\mathfrak{C} \equiv \mathfrak{D}$, for some countable completely representable \mathfrak{D} that satisfies the two conditions in (*). Since $\mathfrak{D} \subseteq_d \mathfrak{CmAt}\mathfrak{D}$, we get the required result, because $\mathfrak{D} \in \mathbf{S}_d \mathbf{Nr}_n \mathbf{CA}_\omega$ and as before $\mathfrak{C} \notin \mathbf{S}_c \mathbf{Nr}_n \mathbf{CA}_{n+3}$ and $\mathfrak{C} \equiv \mathfrak{D}$. For the following part, let \mathbf{L} be as specified and \mathfrak{D} and $\mathfrak{C} = \mathbf{CA}_{\mathbb{Z}, \mathbb{N}}$ be the algebras in the last paragraph. Since an atom structure of an algebra is first order interpretable in the algebra, then we have $\mathfrak{D} \equiv \mathfrak{C} \implies \text{At}\mathfrak{D} \equiv \text{At}\mathfrak{C}$. Furthermore $\text{At}\mathfrak{D} \in \text{At}(\mathbf{Nr}_n \mathbf{CA}_\omega) \subseteq \mathbf{L}$ (as stated above \mathfrak{D} might not be in $\mathbf{Nr}_n \mathbf{CA}_\omega$) and $\text{At}\mathfrak{C} \notin \text{At}(\mathbf{S}_c \mathbf{Nr}_n \mathbf{CA}_{n+3}) \supseteq \mathbf{L}$. The latter follows from the fact that if $\mathfrak{F} \in \mathbf{CA}_n$ is atomic, then $\text{At}\mathfrak{F} \in \text{At}(\mathbf{S}_c \mathbf{Nr}_n \mathbf{CA}_{n+3}) \iff \mathfrak{F} \in \mathbf{S}_c \mathbf{Nr}_n \mathbf{CA}_{n+3}$. We conclude that \mathbf{L} is not elementary.

For the last part, It suffices to consider classes between $\mathbf{Nr}_n \mathbf{CA}_\omega$ and $\mathbf{S}_d \mathbf{Nr}_n \mathbf{CA}_\omega$, because the former class is not elementary [38, Theorem 5.4.1] and as just shown any class between $\mathbf{S}_d \mathbf{Nr}_n \mathbf{CA}_\omega$ and $\mathbf{S}_c \mathbf{Nr}_n \mathbf{CA}_{n+3}$ is not elementary. One implication, namely, \Leftarrow is trivial. For the other less trivial (but still easy) implication, assume for contradiction that there is such a class \mathbf{K} that is elementary. Then $\mathbf{EINr}_n \mathbf{CA}_\omega \subseteq \mathbf{K}$, because \mathbf{K} is elementary. It readily follows that $\mathbf{Nr}_n \mathbf{CA}_\omega \subseteq \mathbf{EINr}_n \mathbf{CA}_\omega \subseteq \mathbf{K} \subseteq \mathbf{S}_d \mathbf{Nr}_n \mathbf{CA}_\omega$, which is impossible by the given assumption that $\mathbf{EINr}_n \mathbf{CA}_\omega \subsetneq \mathbf{S}_d \mathbf{Nr}_n \mathbf{CA}_\omega$. \square

From a result of Venema’s [19, Theorem 2.96] by noting that varieties of \mathbf{CA}_n s are conjugated, we readily conclude from the first item that $\mathbf{SNr}_n \mathbf{CA}_{n+k}$ cannot be axiomatized by Sahlqvist equations.

The next theorem taken from [40] highlights the limitations of the scope of the previous proof on first order definability. These limitations for the relation algebra analogue were crossed in [43], an error that was corrected in [17] by weakening the result in [43].

Theorem 9.5. *Let α be any ordinal > 1 . Then for every infinite cardinal $\kappa \geq |\alpha|$, there exist completely representable algebras $\mathfrak{B}, \mathfrak{A} \in \mathbf{CA}_\alpha$, that are weak set algebras, such that $\mathbf{At}\mathfrak{A} = \mathbf{At}\mathfrak{B}$, $|\mathbf{At}\mathfrak{B}| = |\mathfrak{B}| = \kappa$, $\mathfrak{B} \notin \mathbf{E}|\mathbf{Nr}_\alpha \mathbf{CA}_{\alpha+1}$, $\mathfrak{A} \in \mathbf{Nr}_\alpha \mathbf{CA}_{\alpha+\omega}$, and $\mathbf{EmAt}\mathfrak{B} = \mathfrak{A}$, so that $|\mathfrak{A}| = 2^\kappa$. In particular, $\mathbf{Nr}_\alpha \mathbf{CA}_\beta \subsetneq \mathbf{S}_d \mathbf{Nr}_\alpha \mathbf{CA}_\beta \subseteq \mathbf{S}_c \mathbf{Nr}_\alpha \mathbf{CA}_\beta$.*

Proof. Fix an infinite cardinal $\kappa \geq |\alpha|$. Let FT_α denote the set of all finite transformations on an ordinal α . Assume that $\alpha > 1$. Let \mathfrak{F} be field of characteristic 0 such that $|\mathfrak{F}| = \kappa$, $V = \{s \in {}^\alpha \mathfrak{F} : |\{i \in \alpha : s_i \neq 0\}| < \omega\}$ and let \mathfrak{A} have universe $\wp(V)$ with the usual concrete operations. Then clearly $\wp(V) \in \mathbf{Nr}_\alpha \mathbf{CA}_{\alpha+\omega}$. Let y denote the following α -ary relation: $y = \{s \in V : s_0 + 1 = \sum_{i>0} s_i\}$. Let y_s be the singleton containing s , i.e. $y_s = \{s\}$. Let $\mathfrak{B} = \mathbf{S}g^{\mathfrak{A}}\{y, y_s : s \in y\}$. Clearly $|\mathfrak{B}| = \kappa$. Now \mathfrak{B} and \mathfrak{A} having same top element V , share the same atom structure, namely, the singletons. Thus $\mathbf{EmAt}\mathfrak{B} = \mathfrak{A}$. As proved in [40], we have $\mathfrak{B} \notin \mathbf{E}|\mathbf{Nr}_\alpha \mathbf{CA}_{\alpha+1}$. \square

Corollary 9.6. *Let α be any ordinal > 1 . Then there exists a completely representable $\mathfrak{B} \in \mathbf{CA}_\alpha$ such that $\mathbf{At}\mathfrak{B} \in \mathbf{At}\mathbf{Nr}_\alpha \mathbf{CA}_{\alpha+\omega}$, $\mathbf{EmAt}\mathfrak{B} \in \mathbf{Nr}_\alpha \mathbf{CA}_{\alpha+\omega}$, but $\mathfrak{B} \notin \mathbf{Nr}_\alpha \mathbf{CA}_{\alpha+1} (\supseteq \mathbf{Nr}_\alpha \mathbf{CA}_{\alpha+\omega})$.*

We conclude from the previous corollary that we cannot lift the result proved in theorem 9.4 for the class \mathbf{L} to the algebra level deleting \mathbf{At} . The result proved for atom structures is stronger than that proved on the level of algebras. The condition imposed in the last part of the theorem is both sufficient and necessary for lifting the result to the algebra level.

What was proved in [17] is the Ra analogue of the following which follows from item (2) of theorem 9.4.

Corollary 9.7. *Any class between \mathbf{CRCA}_n and $\mathbf{S}_c \mathbf{Nr}_n \mathbf{CA}_{n+3}$ is not elementary. In particular, any class between $\mathbf{SNr}_n \mathbf{CA}_\omega \cap \mathbf{At}$ and $\mathbf{S}_c \mathbf{Nr}_n \mathbf{CA}_{n+3}$ is not elementary.*

For relation algebras a yet strictly stronger result than that that proved in [17], but weaker than that alledged in [43] can be proved:

Corollary 9.8. *Any class between $\mathbf{RRA} \cap \mathbf{S}_d \mathbf{RaCA}_\omega$ and $\mathbf{S}_c \mathbf{RaCA}_k$, $k \geq 6$, is not elementary.*

The strictness of the inclusion $\mathbf{S}_d \mathbf{RaCA}_\omega \subsetneq \mathbf{S}_c \mathbf{RaCA}_\omega$ is proved in [43]. However, in corollary 9.8 we do not know whether we can replace $\mathbf{S}_d \mathbf{RaCA}_\omega$ by \mathbf{RaCA}_ω , which we will succeed to do in the \mathbf{CA} case in a moment. As a matter of fact, we do not know in the first place whether the last two classes are distinct or not, i.e whether the inclusion $\mathbf{RaCA}_\omega \subseteq \mathbf{S}_d \mathbf{RaCA}_\omega$ is strict or not. However, by example 9.5 we know that the \mathbf{CA} analogue of this inclusion is strict; the strictness witnessed the algebra denoted by \mathfrak{B} in *op.cit.*

To summarize: Baring in mind the following facts:

- $\mathbf{Nr}_n \mathbf{CA}_\omega \subsetneq \mathbf{S}_d \mathbf{Nr}_n \mathbf{CA}_\omega \subsetneq \mathbf{S}_c \mathbf{Nr}_n \mathbf{CA}_\omega$ by example 9.5, and the construction of \mathfrak{A} and \mathfrak{B} used in the proof of item (1) of the forthcoming theorem 9.9,
- that neither of the classes \mathbf{CRCA}_n and $\mathbf{S}_d \mathbf{Nr}_n \mathbf{CA}_\omega \cap \mathbf{At}$ is contained in the other by the same algebras used in item (1) of theorem 9.9, and the construction in [34]. From the construction in [34], it can be directly distilled, that for every infinite cardinal κ , there an atomic algebra $\mathfrak{B} \in \mathbf{Nr}_n \mathbf{CA}_\omega \sim \mathbf{CRCA}_n$ having 2^κ many atoms.

- that neither of the classes CRCA_n and $\text{Nr}_n\text{CA}_\omega \cap \mathbf{At}$ are contained in each other by the same aforementioned references in the previous item,
- and finally that $\mathbf{S}_d\text{Nr}_n\text{CA}_\omega \cap \text{CRCA}_n \subsetneq \mathbf{S}_c\text{Nr}_n\text{CA}_\omega \cap \text{CRCA}_n$, with \mathfrak{B} used in item (1) of theorem 9.9 witnessing the strictness of the inclusion,

we get that item (2) of theorem 9.4 is stronger than corollary 9.7, that item (2) of the next theorem is stronger than item (2) of theorem 9.4 too, and lastly that item (1) in the following theorem is different than the latter two results.

Now we prove the *strictly* stronger result from that proved in tem (2) of theorem 9.4, obatined by repalcing $\mathbf{S}_d\text{Nr}_n\text{CA}_\omega$ by the (smaller) class $\text{Nr}_n\text{CA}_\omega$. using the intervention of another construction.

Theorem 9.9. 1. Any class between $\text{Nr}_n\text{CA}_\omega \cap \text{CRCA}_n$ and $\mathbf{S}_d\text{Nr}_n\text{CA}_{n+1}$ is not elementary

2. Any class between $\text{Nr}_n\text{CA}_\omega \cap \text{CRCA}_n$ and $\mathbf{S}_c\text{Nr}_n\text{CA}_{n+3}$ is not elementary

Proof. For item (1): We slightly modify the construction in [38, Lemma 5.1.3, Theorem 5.1.4] lifted to any finite $n > 2$. The algebras \mathfrak{A} and \mathfrak{B} constructed in *op.cit* satisfy that $\mathfrak{A} \in \text{Nr}_n\text{CA}_\omega$, $\mathfrak{B} \notin \text{Nr}_n\text{CA}_{n+1}$ and $\mathfrak{A} \equiv \mathfrak{B}$. As they stand, \mathfrak{A} and \mathfrak{B} are not atomic, but it can be fixed that they are atomic, giving the same result, by interpreting the uncountably many n -ary relations in the signature of \mathbf{M} defined in [38, Lemma 5.1.3] for $n = 3$, which is the base of \mathfrak{A} and \mathfrak{B} to be *disjoint* in \mathbf{M} , not just distinct. In fact the construction is presented in this way in [31].

Let us see show why. We work with $2 < n < \omega$ instead of only $n = 3$. The proof presented in *op.cit* lifts verbatim to any such n . Let $u \in {}^n n$. Write $\mathbf{1}_u$ for $\chi_u^{\mathbf{M}}$ (denoted by $\mathbf{1}_u$ (for $n = 3$) in [38, Theorem 5.1.4].) We denote by \mathfrak{A}_u the Boolean algebra $\mathfrak{A}_{\mathbf{1}_u} \mathfrak{A} = \{x \in \mathfrak{A} : x \leq \mathbf{1}_u\}$ and similarly for \mathfrak{B} , writing \mathfrak{B}_u short hand for the Boolean algebra $\mathfrak{A}_{\mathbf{1}_u} \mathfrak{B} = \{x \in \mathfrak{B} : x \leq \mathbf{1}_u\}$. Using that \mathbf{M} has quantifier elimination we get, using the same argument in *op.cit* that $\mathfrak{A} \in \text{Nr}_n\text{CA}_\omega$. The property that $\mathfrak{B} \notin \text{Nr}_n\text{CA}_{n+1}$ is also still maintained. To see why consider the substitution operator ${}_n s(0, 1)$ (using one spare dimension) as defined in the proof of [38, Theorem 5.1.4].

Assume for contradiction that $\mathfrak{B} = \text{Nr}_n \mathfrak{C}$, with $\mathfrak{C} \in \text{CA}_{n+1}$. Let $u = (1, 0, 2, \dots, n-1)$. Then $\mathfrak{A}_u = \mathfrak{B}_u$ and so $|\mathfrak{B}_u| > \omega$. The term ${}_n s(0, 1)$ acts like a substitution operator corresponding to the transposition $[0, 1]$; it ‘swaps’ the first two co-ordinates. Now one can show that ${}_n s(0, 1)^{\mathfrak{C}} \mathfrak{B}_u \subseteq \mathfrak{B}_{[0, 1] \circ u} = \mathfrak{B}_{Id}$, so $|{}_n s(0, 1)^{\mathfrak{C}} \mathfrak{B}_u|$ is countable because \mathfrak{B}_{Id} was forced by construction to be countable. But ${}_n s(0, 1)$ is a Boolean automorphism with inverse ${}_n s(1, 0)$, so that $|\mathfrak{B}_{Id}| = |{}_n s(0, 1)^{\mathfrak{C}} \mathfrak{B}_u| > \omega$, contradiction. One proves that $\mathfrak{A} \equiv \mathfrak{B}$ exactly like in [38].

Now we show that \mathfrak{B} is actually outside the bigger class $\mathbf{S}_d\text{Nr}_n\text{CA}_{n+1}$. Take the cardinality κ specifying the signature of \mathbf{M} to be 2^{2^ω} and assume for contradiction that $\mathfrak{B} \in \mathbf{S}_d\text{Nr}_n\text{CA}_{n+1} \cap \mathbf{At}$. Then $\mathfrak{B} \subseteq_d \mathfrak{Nr}_n \mathfrak{D}$, for some $\mathfrak{D} \in \text{CA}_{n+1}$ and $\mathfrak{Nr}_n \mathfrak{D}$ is atomic. For brevity, let $\mathfrak{C} = \mathfrak{Nr}_n \mathfrak{D}$. Then $\mathfrak{B}_{Id} \subseteq_d \mathfrak{A}_{Id} \mathfrak{C}$; the last algebra is the Boolean algebra with universe $\{x \in \mathfrak{C} : x \leq Id\}$. Since \mathfrak{C} is atomic, $\mathfrak{A}_{Id} \mathfrak{C}$ is also atomic. Using the same reasoning as above, we get that $|\mathfrak{A}_{Id} \mathfrak{C}| > 2^\omega$ (since $\mathfrak{C} \in \text{Nr}_n\text{CA}_{n+1}$). By the choice of κ , we get that $|\text{At} \mathfrak{A}_{Id} \mathfrak{C}| > \omega$. By $\mathfrak{B} \subseteq_d \mathfrak{C}$, we get that $\mathfrak{B}_{Id} \subseteq_d \mathfrak{A}_{Id} \mathfrak{C}$ and that

$\text{At}\mathfrak{R}l_{Id}\mathfrak{C} \subseteq \text{At}\mathfrak{B}_{Id}$, so $|\text{At}\mathfrak{B}_{Id}| \geq |\text{At}\mathfrak{R}l_{Id}\mathfrak{C}| > \omega$. But by the construction of \mathfrak{B} , we have $|\mathfrak{B}_{Id}| = |\text{At}\mathfrak{B}_{Id}| = \omega$, which is a contradiction and we are done.

For the second required, namely, item (2): We need to remove the \mathbf{S}_d from item (2) of theorem 9.4. Example 9.5 alerts us to the fact that we are not done yet. We still have some work to do. For this purpose, we need the intervention of the construction used in item (1). We use the (modified) algebras \mathfrak{A} and \mathfrak{B} that we know satisfy: $\mathfrak{A} \in \text{Nr}_n\text{CA}_\omega$, $\mathfrak{B} \notin \text{Nr}_n\text{CA}_{n+1}$, $\mathfrak{A} \equiv \mathfrak{B}$ and both \mathfrak{A} and \mathfrak{B} are atomic.

In item (2) of theorem 9.4, we have excluded any first order definable class between the two classes $\mathbf{S}_d\text{Nr}_n\text{CA}_\omega \cap \text{CRCA}_n$ and $\mathbf{S}_c\text{Nr}_n\text{CA}_{n+3}$. So hoping for a contradiction, we can only assume that there is a class \mathbf{M} between $\text{Nr}_n\text{CA}_\omega \cap \text{CRCA}_n$ and $\mathbf{S}_d\text{Nr}_n\text{CA}_\omega \cap \text{CRCA}_n$ that is first order definable. Then $\mathbf{EI}(\text{Nr}_n\text{CA}_\omega \cap \text{CRCA}_n) \subseteq \mathbf{M} \subseteq \mathbf{S}_d\text{Nr}_n\text{CA}_\omega \cap \text{CRCA}_n$. We have, $\mathfrak{B} \equiv \mathfrak{A}$, and $\mathfrak{A} \in \text{Nr}_n\text{CA}_\omega \cap \text{CRCA}_n$, hence $\mathfrak{B} \in \mathbf{EI}(\text{Nr}_n\text{CA}_\omega \cap \text{CRCA}_n) \subseteq \mathbf{S}_d\text{Nr}_n\text{CA}_\omega \cap \text{CRCA}_n$. But in item (1), we showed that \mathfrak{B} is in fact outside $\mathbf{S}_d\text{Nr}_n\text{CA}_{n+1} \supseteq \mathbf{S}_d\text{Nr}_n\text{CA}_\omega \supseteq \mathbf{S}_d\text{Nr}_n\text{CA}_\omega \cap \mathbf{At} \supseteq \mathbf{S}_d\text{Nr}_n\text{CA}_\omega \cap \text{CRCA}_n$, getting the hoped for contradiction, and consequently the required. \square

Corollary 9.10. *Let $2 < n < \omega$ and $k \geq 3$. Then the following classes, together with the intersection of any two of them, the last four taken at the same k , are not elementary: CRCA_n [18], $\text{Nr}_n\text{CA}_{n+k}$ [38, Theorem 5.4.1], $\mathbf{S}_d\text{Nr}_n\text{CA}_{n+k}$, $\mathbf{S}_c\text{Nr}_n\text{CA}_{n+k}$ and the class of algebras having complete k -flat representations [39].*

As indicated above, we do not know whether the Ra analogous of item (2) proved in theorem 9.9 holds or not. But the closely related following result is proved in [35] using instead Monk-like algebras. The integral relation algebra (in which Id is an atom) defined next by listing its forbidden triples, will be used in the proof of the next theorem.

Example 9.11. Take \mathfrak{R} to be a symmetric, atomic relation algebra with atoms

$$\text{Id}, r(i), y(i), b(i) : i < \omega.$$

Non-identity atoms have colours, r is red, b is blue, and y is yellow. All atoms are self-converse. Composition of atoms is defined by listing the forbidden triples. The forbidden triples are (Peircean transforms) or permutations of (Id, x, y) for $x \neq y$, and

$$(r(i), r(i), r(j)), (y(i), y(i), y(j)), (b(i), b(i), b(j))) \quad i \leq j < \omega$$

\mathfrak{R} is the complex algebra over this atom structure.

Let α be an ordinal. \mathfrak{R}^α is obtained from \mathfrak{R} by splitting the atom $r(0)$ into α parts $r^k(0) : k < \alpha$ and then taking the full complex algebra. In more detail, we put red atoms $r^k(0)$ for $k < \alpha$. In the altered algebra the forbidden triples are $(y(i), y(i), y(j)), (b(i), b(i), b(j)))$, $i \leq j < \omega$, $(r(i), r(i), r(j))$, $0 < i \leq j < \omega$, $(r^k(0), r^l(0), r(j))$, $0 < j < \omega, k, l < \alpha$, $(r^k(0), r^l(0), r^m(0))$, $k, l, m < \alpha$. These algebras were used in [34] to show that $\mathfrak{R}\mathbf{aCA}_k$ for all $k \geq 5$ is not elementary.

Theorem 9.12. *Any class between $\mathfrak{R}\mathbf{aCA}_\omega$ and $\mathfrak{R}\mathbf{aCA}_5$, as well as the class CRR , is not closed under $\equiv_{\infty, \omega}$.*

Proof. In \mathfrak{R}^α , we use the following abbreviations: $r(0) = \sum_{k < \alpha} r^k(0)$, $r = \sum_{i < \omega} r(i)$, $y = \sum_{i < \omega} y(i)$ and $b = \sum_{i < \omega} b(i)$. These suprema exist because they are taken in the

complex algebras which are complete. The *index* of $r(i), y(i)$ and $b(i)$ is i and the index of $r^k(0)$ is also 0. Now let $\mathfrak{B} = \mathfrak{R}^\omega$ and $\mathfrak{A} = \mathfrak{R}^n$. We claim that $\mathfrak{B} \in \mathfrak{RaCA}_\omega$ and $\mathfrak{A} \equiv \mathfrak{B}$. For the first required, it is shown in [34] that \mathfrak{B} has a cylindric basis by exhibiting a winning strategy for \exists in the the cylindric-basis game, which is a simpler version of the hyperbasis game [19, Definition 12.26]. Now, let \mathcal{H} be an ω -dimensional cylindric basis for \mathfrak{B} . Then $\mathfrak{CaH} \in \mathfrak{CA}_\omega$. Consider the cylindric algebra $\mathfrak{C} = \mathfrak{Sg}^{\mathfrak{CaH}} \mathfrak{B}$, the subalgebra of \mathfrak{CaH} generated by \mathfrak{B} . In principal, new two dimensional elements that were not originally in \mathfrak{B} , can be created in \mathfrak{C} using the spare dimensions in $\mathfrak{Ca}(\mathcal{H})$. But in fact \mathfrak{B} exhausts the 2-dimensional elements of \mathfrak{RaC} , more concisely, we have $\mathfrak{B} = \mathfrak{RaC}$ [35]. Like the proof of theorem 9.9, we can show that $\mathfrak{A} \equiv_{\infty, \omega} \mathfrak{B}$ by replacing $\mathbf{1}_{\text{Id}}$ by the newly splitted $r(0)$. We have proved that $\mathfrak{B} \in \mathfrak{RaCA}_\omega$ and $\mathfrak{A} \equiv \mathfrak{B}$. In [35], it is proved that $\mathfrak{A} \notin \mathfrak{RaCA}_5$. So we get the first required, namely, that any class \mathbf{K} , such that $\mathfrak{RaCA}_\omega \subseteq \mathbf{K} \subseteq \mathfrak{RaCA}_5$ is not closed under $\equiv_{\infty, \omega}$.

Now we show that CRRA is not closed under $\equiv_{\infty, \omega}$, strengthening the result in [18] that only shows that CRRA is not closed under elementary equivalence proving the remaining required. Since $\mathfrak{B} \in \mathfrak{RaCA}_\omega$ has countably many atoms, then \mathfrak{B} is completely representable [43, Theorem 29]. For this purpose, we show that \mathfrak{A} is not completely representable. We work with the term algebra, \mathfrak{TmAtA} , since the latter is completely representable \iff the complex algebra is. Let $r = \{r(i) : 1 \leq i < \omega\} \cup \{r^k(0) : k < 2^{\aleph_0}\}$, $y = \{y(i) : i \in \omega\}$, $b^+ = \{b(i) : i \in \omega\}$. It is not hard to check every element of $\mathfrak{TmAtA} \subseteq \wp(\text{AtA})$ has the form $F \cup R_0 \cup B_0 \cup Y_0$, where F is a finite set of atoms, R_0 is either empty or a co-finite subset of r , B_0 is either empty or a co-finite subset of b , and Y_0 is either empty or a co-finite subset of y . We show that the existence of a complete representation necessarily forces a monochromatic triangle, that we avoided at the start when defining \mathfrak{A} . Let x, y be points in the representation with $M \models y(0)(x, y)$. For each $i < 2^{\aleph_0}$, there is a point $z_i \in M$ such that $M \models \text{red}(x, z_i) \wedge y(0)(z_i, y)$ (some red $\text{red} \in r$). Let $Z = \{z_i : i < 2^{\aleph_0}\}$. Within Z each edge is labelled by one of the ω atoms in y^+ or b^+ . The Erdos-Rado theorem forces the existence of three points $z^1, z^2, z^3 \in Z$ such that $M \models y(j)(z^1, z^2) \wedge y(j)(z^2, z^3) \wedge y(j)(z^3, z^1)$, for some single $j < \omega$ or three points $z^1, z^2, z^3 \in Z$ such that $M \models b(l)(z^1, z^2) \wedge b(l)(z^2, z^3) \wedge b(l)(z^3, z^1)$, for some single $l < \omega$. This contradicts the definition of composition in \mathfrak{A} (since we avoided monochromatic triangles). We have proved that CRRA is not closed under $\equiv_{\infty, \omega}$, since $\mathfrak{A} \equiv_{\infty, \omega} \mathfrak{B}$, \mathfrak{A} is not completely representable, but \mathfrak{B} is completely representable. \square

Remark: Fix $2 < n < \omega$. The algebra $\mathfrak{A} \in \text{Nr}_n \mathfrak{CA}_\omega$ used in theorem 9.9 can be viewed as splitting the atoms of the atom structure $\mathbf{At} = ({}^n n, \equiv_i, D_{ij})_{i, j < n}$ each to κ -many atoms so \mathfrak{A} can be denoted $\text{split}(\mathbf{1}_{\text{Id}}, \mathbf{At}, \kappa)$ (κ an uncountable cardinal). The algebra $\mathfrak{B} \notin \text{Nr}_n \mathfrak{CA}_{n+1}$ can be viewed as splitting the same atom structure, each atom – except for one atom that is split into countably many atoms – is also split into κ -many atoms, so \mathfrak{B} can be denoted by $\text{split}(\mathbf{1}_{\text{Id}}, \mathbf{At}, \omega)$. By the same token, for an ordinal α , \mathfrak{R}^α can be denoted by $\text{split}(r(0), \mathfrak{R}, \alpha)$ short for splitting (the red atom) $r(0)$ into α parts.

Corollary 9.13. *Let $k \geq 5$. Then the classes CRRA, RaCA_k , $\text{S}_d \text{RaCA}_k$ and $\text{S}_c \text{RaCA}_k$ are not elementary [18, 35, 43]. The first two classes are not closed under $\equiv_{\infty, \omega}$, but are closed under ultraproducts. Furthermore, $\mathfrak{RaCA}_\omega \subseteq \text{S}_d \mathfrak{RaCA}_\omega \subsetneq \text{S}_c \mathfrak{RaCA}_\omega$.*

Proof. The first two classes are closed under ultraproducts because they are psuedo-elementary (reducts of elementary classes), cf. [19, Item (2), p. 279], [43, Theorem 21].

Proving the strictness of the last inclusion can be easily distilled from the proof of [43, Theorem 36]. The rest of the alleged statements can also be found in [43]. \square

For a class \mathbf{K} of BAOs, recall that $\mathbf{K} \cap \mathbf{At}$ denotes the class of atomic algebras in \mathbf{K} . Let $\mathbf{FRRA} = \{\mathfrak{A} \in \mathbf{RRA} : A = \wp(^2U) \text{ some non-empty set } U\}$. We use the relation algebra analogue of lemma 9.2 proved in [43]. Following the notation in *top.cit*, we denote the the \mathfrak{Ra} analogue of the game \mathbf{G}^m by F^m which is like the usual atomic game G_r^m [19], when $r = \omega$, and \forall is allowed to reuse the m nodes in play.

From the blow up and blur construction used in [7] one can show that the existence a finite relation algebra with an n -blur and no infinite m -dimensional hyperbasis with $2 < n < m \leq \omega$ implies that $\mathbf{SNr}_n \mathbf{CA}_m$ is not atom-canonical. Assume that \mathfrak{R} is such a relation algebra. Take $\mathfrak{B} = \mathbf{Bb}_n(\mathfrak{R}, J, E)$ with last notation as in [7, Top of p. 78]. Then $\mathfrak{B} \in \mathbf{RCA}_n$. We claim that $\mathfrak{C} = \mathbf{EmAt}\mathfrak{B} \notin \mathbf{SNr}_n \mathbf{CA}_m$. Suppose for contradiction that $\mathfrak{C} \subseteq \mathbf{Nr}_n \mathfrak{D}$, where \mathfrak{D} is atomic, with $\mathfrak{D} \in \mathbf{CA}_m$. Then \mathfrak{C} has a (necessarily infinite m -flat representation), hence $\mathfrak{Ra}\mathfrak{C}$ has an infinite m -flat representation as an RA. But \mathfrak{R} embeds into $\mathbf{EmAt}(\mathbf{Bb}(\mathfrak{R}, J, E))$ which, in turn, embeds into $\mathfrak{Ra}\mathfrak{C}$, so \mathfrak{R} has an infinite m -flat representation. Hence \mathfrak{R} has a m -dimensional infinite hyperbases which contradicts the hypothesis.

We know that such algebras exist, having even *strong l -blurs* for any pre-assigned $n \leq l < \omega$ when $m = \omega$, identifying the existence of an ω -dimensional infinite hyperbasis with the existence of a representation on an infinite base. Such algebras are the Maddux algebras denoted in [7, Lemma 5] by $\mathfrak{E}_k(2, 3)$ with k the finite number of non-identity atoms depending recursively on l . Here by a strong l -blur, we understand an l blur (J, E) (J is the set of blure and E is the index blur) as in [7, Definition 3.1] satisfying using the notation in *op.cit*: $(\forall V_1, \dots V_n, W_2, \dots W_n \in J)(\forall T \in J)(\forall 2 \leq i \leq n) \text{safe}(V_i, W_i, T)$ that is, there is for $v \in V_i$, $w \in W_i$ and $t \in T$, $v; w \leq t$. This last condition formulated as $(J5)_n$ on [7, p.79], is obtained from $(J4)_n$ formulated in the definition of an l -blur [7, p.72], by replacing \exists by \forall plainly giving a stronger condition.

The following theorem on non-atom canonicity was proved in [36] using Monk-like algebras.

Theorem 9.14. *Let $2 < n < \omega$. Then there exists an atomic countable relation algebra \mathfrak{R} , such that $\mathbf{Mat}_n(\mathbf{At}\mathfrak{R})$ forms an n -dimensional cylindric basis, $\mathfrak{A} = \mathbf{ImMat}_n(\mathbf{At}\mathfrak{R}) \in \mathbf{RQEA}_n$, while even the diagonal free reduct of the Dedekind-MacNeille completion of \mathfrak{A} , namely, $\mathbf{EmMat}_n(\mathbf{At}\mathfrak{R})$ is not representable. In particular, $\mathbf{Mat}_n(\mathbf{At}\mathfrak{R})$ is a weakly, but not strongly representable \mathbf{QEA}_n atom structure.*

Proof. Let \mathfrak{G} be a graph. Let ρ be a ‘shade of red’; we assume that $\rho \notin \mathfrak{G}$. Let L^+ be the signature consisting of the binary relation symbols (a, i) , for each $a \in \mathfrak{G} \cup \{\rho\}$ and $i < n$. Let T denote the following (Monk) theory in this signature: $M \models T \iff$ for all $a, b \in M$, there is a unique $p \in (\mathfrak{G} \cup \{\rho\}) \times n$, such that $(a, b) \in p$ and if $M \models (a, i)(x, y) \wedge (b, j)(y, z) \wedge (c, k)(x, z)$, $x, y, z \in M$, then $|i, j, k| > 1$, or $a, b, c \in \mathfrak{G}$ and $\{a, b, c\}$ has at least one edge of \mathfrak{G} , or exactly one of a, b, c – say, a – is ρ , and bc is an edge of \mathfrak{G} , or two or more of a, b, c are ρ .

We denote the class of models of T which can be seen as coloured undirected graphs (not necessarily complete) with labels coming from $(\mathfrak{G} \cup \{\rho\}) \times n$ by \mathbf{G} .

Now specify \mathfrak{G} to be either:

- the graph with nodes \mathbb{N} and edge relation E defined by $(i, j) \in E$ if $0 < |i - j| < N$, where $N \geq n(n - 1)/2$ is a positive number.
- or the ω disjoint union of N cliques, same N .

In both cases the countably infinite graphs contain infinitely many N cliques. In the first they overlap, in the second they do not. One shows that there is a countable (n -homogeneous) coloured graph (model) $M \in \mathbf{G}$ with the following property [24, Proposition 2.6]:

If $\Delta \subseteq \Delta' \in \mathbf{G}$, $|\Delta'| \leq n$, and $\theta : \Delta \rightarrow M$ is an embedding, then θ extends to an embedding $\theta' : \Delta' \rightarrow M$.

Here the choice of $N \geq n(n-1)/n$ is not haphazard it bounds the number of edges of any graph Δ of size $\leq n$. This is crucial to show that for any permutation χ of $\omega \cup \{\rho\}$, Θ^χ is an n -back-and-forth system on M [36], so that the countable atomic set algebra \mathfrak{A} based on M whose top element is obtained from ${}^n M$ by discarding assignments whose edges are labelled by one of n -shaded of reds $((\rho, i) : i < n)$ forming $W \subsetneq {}^n M$, is classically representable. The classical semantics of $L_{\omega, \omega}$ formulas and relativized semantics (restricting assignments to W), coincide, so that \mathfrak{A} is isomorphic to a set algebra with top element ${}^n M$.

Consider the following relation algebra atom structure $\alpha(\mathfrak{G}) = (\{\text{Id}\} \cup (\mathfrak{G} \times n), R_{\text{Id}}, \check{R}, R;)$, where: The only identity atom is Id . All atoms are self converse, so $\check{R} = \{(a, a) : a \text{ an atom}\}$. The colour of an atom $(a, i) \in \mathfrak{G} \times n$ is i . The identity Id has no colour. A triple (a, b, c) of atoms in $\alpha(\mathfrak{G})$ is consistent if $R; (a, b, c)$ holds ($R;$ is the accessibility relation corresponding to composition). Then the consistent triples are (a, b, c) where: One of a, b, c is Id and the other two are equal, or none of a, b, c is Id and they do not all have the same colour, or $a = (a', i), b = (b', i)$ and $c = (c', i)$ for some $i < n$ and $a', b', c' \in \mathfrak{G}$, and there exists at least one graph edge of G in $\{a', b', c'\}$.

\mathfrak{C} is not representable because $\mathfrak{Cm}(\alpha(\mathfrak{G}))$ is not representable and $\text{Mat}_n(\alpha(\mathfrak{G})) \cong \text{At}\mathfrak{A}$. To see why, for each $m \in \text{Mat}_n(\alpha(\mathfrak{G}))$, let $\alpha_m = \bigwedge_{i, j < n} \alpha_{ij}$. Here α_{ij} is $x_i = x_j$ if $m_{ij} = \text{Id}$ and $R(x_i, x_j)$ otherwise, where $R = m_{ij} \in L$. Then the map $(m \mapsto \alpha_m^W)_{m \in \text{Mat}_n(\alpha(\mathfrak{G}))}$ is a well - defined isomorphism of n -dimensional cylindric algebra atom structures. Non-representability follows from the fact that \mathfrak{G} is a bad graph, that is, $\chi(\mathfrak{G}) = N < \infty$ [19, Definition 14.10, Theorem 14.11]. The relation algebra atom structure specified above is exactly like the one in Definition 14.10 in *op.cit*, except that we have n colours rather than just three. \square

Now using the construction in [36, 7] on ‘non atom-canonicity’, we prove ‘non-finite axiomatizability results for relation and cylindric algebra. Let LCA_n denote the elementary class of RCA_n s satisfying the Lyndon conditions. We stipulate that $\mathfrak{A} \in \text{LCA}_n \iff \mathfrak{A}$ is atomic and $\text{At}\mathfrak{A}$ satisfies the Lyndon conditions [21, Definition 3.5.1]. In the same sense, let $\text{LCRA}(\subseteq \text{RRA})$ denote the elementary class of RAs satisfying the Lyndon conditions [19, Definition pp. 337]. We now use the above construction proving non-atom canonicity to prove non-finite axiomatizability. We use ‘bad’ non-representable Monk-like algebras converging to a ‘good’ representable one. In the process, we recover the results of Monk and Maddux on non-finite axiomatizability of both RCA_n ($2 < n < \omega$) and RRA .

Let $2 < n < \omega$. Then LCRA and LCA_n are not finitely axiomatizable.

First proof: Let \mathfrak{A}_l be the atomic RCA_n constructed from \mathfrak{G}_l , $l \in \omega$ where \mathfrak{G}_l which is a disjoint countable union of N_l cliques, such that for $i < j \in \omega$, $n(n-1)/n \leq N_i < N_j$. Then

$\mathfrak{Cm}\mathfrak{A}_l$ with \mathfrak{A}_l based on \mathfrak{G}_l , as constructed in the proof of theorem 9.14 is not representable. So $(\mathfrak{Cm}(\mathfrak{A}_l) : l \in \omega)$ is a sequence of non-representable algebras, whose ultraproduct \mathfrak{B} , being based on the ultraproduct of graphs having arbitrarily large chromatic number, will have an infinite clique, and so \mathfrak{B} will be completely representable [21, Theorem 3.6.11]. Likewise, the sequence $(\mathfrak{Im}(\mathfrak{A}_l) : l \in \omega)$ is a sequence of representable, but not completely representable algebras, whose ultraproduct is completely representable. The same holds for the sequence of relation algebras $(\mathfrak{R}_l : l \in \omega)$ constructed as in [36] for which $\mathfrak{ImAt}\mathfrak{A}_l \cong \text{Mat}_n \text{At}\mathfrak{R}_l$.

Second proof: For each $2 < n \leq l < \omega$, let \mathfrak{R}_l be the finite Maddux algebra $\mathfrak{C}_{f(l)}(2, 3)$ used in [7] with strong l -blur (J_l, E_l) and $f(l) \geq l$ denoted by $l < k < \omega$ in [7, Lemma 5]. The relation algebra \mathfrak{R}_l has no representations on an infinite base. Let $\mathcal{R}_l = \text{Bb}(\mathfrak{R}_l, J_l, E_l) \in \text{RRA}$ with atom structure At obtained by blowing up and blurring \mathfrak{R}_l (with underlying set denoted by At on [7, p.73]), and let $\mathfrak{A}_l = \text{Nr}_n \text{Bb}_l(\mathfrak{R}_l, J_l, E_l)$ as defined on [7, Top of p. 78]; then $\mathfrak{A}_l \in \text{RCA}_n$. Then $(\text{At}\mathcal{R}_l : l \in \omega \sim n)$, and $(\text{At}\mathfrak{A}_l : l \in \omega \sim n)$ are sequences of weakly representable atom structures that are not strongly representable with a completely representable ultraproduct. The (complex algebra) sequences $(\mathfrak{CmAt}\mathcal{R}_l : l \in \omega \sim n)$, $(\mathfrak{CmAt}\mathfrak{A}_l : l \in \omega \sim n)$ are typical examples of what Hirsch and Hodkinson call ‘bad Monk (non-representable) algebras’ converging to ‘good (representable) one, namely their (non-trivial) ultraproduct. Observe that for $2 < n \leq k < m < \omega$, $\mathfrak{A}_k = \text{Nr}_k \mathfrak{A}_m$. Such sequences witness the non-finite axiomatizability of the class representable algebras and the elementary closure of the class completely representable ones, namely, the class of algebras satisfying the Lyndon conditions. This recovers Monk’s and Maddux’s classical results on non-finite axiomatizability of RRAs and RCA_n s since algebras considered are generated by a single 2-dimensional elements.

9.3 Rainbows versus Monk-like algebras, pros and cons

The model-theoretic ideas used in the theorems 9.4 and the construction in [36] proving theorem 9.14 are quite close but only superficially. In the overall structure; they follow closely the model-theoretic framework in [24]. In both cases, we have finitely many shades of red outside a Monk-like and rainbow signature, that were used as labels to construct an n -homogeneous model M in the expanded signature. Though the shades of reds are *outside* the signature, they were used as labels during an ω -rounded game played on labelled finite graphs—which can be seen as finite models in the extended signature having size $\leq n$ —in which \exists had a winning strategy, enabling her to construct the required M as a countable limit of the finite graphs played during the game. The construction, in both cases, entailed that any subgraph (substructure) of M of size $\leq n$, is independent of its location in M ; it is uniquely determined by its isomorphism type.

A relativized set algebra \mathfrak{A} based on M was constructed by discarding all assignments whose edges are labelled by these shades of reds, getting a set of n -ary sequences $W \subseteq {}^n M$. This W is definable in ${}^n M$ by an $L_{\infty, \omega}$ formula and the semantics with respect to W coincides with classical Tarskian semantics (on ${}^n M$) for formulas of the signature taken in L_n (but not for formulas taken in $L_{\infty, \omega}^n$). This was proved in both cases using certain n back-and-forth systems, thus \mathfrak{A} is representable classically, in fact it (is isomorphic to a set algebra that) has base M . *The heart and soul of both proofs is to replace the reds labels by suitable non-red binary relation symbols within an n back-and-forth system, so*

that one can adjust that the system maps a tuple $\bar{b} \in {}^n M \setminus W$ to a tuple $\bar{c} \in W$ and this will preserve any formula containing the non-red symbols that are ‘moved’ by the system. In fact, all injective maps of size $\leq n$ defined on M modulo an appropriate permutation of the reds will form an n back-and-forth system.

This set algebra \mathfrak{A} was further proved to be atomic, countable, and simple (with top element ${}^n M$). Surjections into the subgraphs of size $\leq n$ of M whose edges are not labelled by any shade of red are the atoms of \mathfrak{A} and its Dedekind-MacNeille completion, $\mathfrak{C} = \mathfrak{CmAt}\mathfrak{A}$. Both algebras have top element W , but $\mathfrak{CmAt}\mathfrak{A}$ is not in $\mathbf{SNr}_n \mathbf{CA}_{n+3}$ in case of the rainbow construction, least representable, and it is (only) not representable in the case of the Monk-like algebra. In case of both constructions ‘the shades of red’ – which can be intrinsically identified with non-principal ultrafilters in \mathfrak{A} , were used as colours, together with the principal ultrafilters to represent completely \mathfrak{A}^+ , inducing a representation of \mathfrak{A} . Non-representability of $\mathfrak{CmAt}\mathfrak{A}$ in the Monk case, used Ramsey’s theory. The non neat-embeddability of $\mathfrak{CmAt}\mathfrak{A}$ in the rainbow case, used *the finite number of greens*, that gave us information on when $\mathfrak{CmAt}\mathfrak{A}$ ‘stops to be representable.’ The reds in both cases has to do with representing \mathfrak{A} . The model theory used for both constructions is almost identical.

Nevertheless, from the algebraic point of view, there is a crucial difference. The non-representability of $\mathfrak{CmAt}\mathfrak{A}$ was tested by a game between the two players \forall and \exists . The winning strategy’s of the two players are independent, this is reflected by the fact that we have two ‘independent parameters’ G (the greens) and R (the reds) that are more were finite irreflexive complete graphs. In Monk-like constructions like the one used in [36] to show that \mathbf{RCA}_n is not atom-canonical by constructing a countable atomic \mathfrak{A} the non representability of $\mathfrak{CmAt}\mathfrak{A}$ was also tested by a game between \exists and \forall . But in *op.cit* winning strategy’s are interlinked, one operates through the other; hence only one parameter is the source of colours. This parameter is a graph \mathfrak{G} which is a countable disjoint union of N cliques where $N \geq n(n-1)/2$. Non-representability of the complex algebra $\mathfrak{CmAt}\mathfrak{A}$ in this case depends only on the *chromatic number of \mathfrak{G}* , via an application of Ramsey’s theorem; this number is $< \omega$; in fact it is N .

In both cases two players operate using ‘cardinality of a labelled graph’. \forall trying to make this graph too large for \exists to cope, colouring some of its edges suitably. For the rainbow case, it is a red clique formed during the play as we have seen in the model theoretic proof of theorem 9.4. It might be clear in both cases (rainbow and Monk-like algebras), to see that \exists cannot win the infinite game, but what might not be clear is *when does this happens; we know it eventually happen in finitely many round, but till which round \exists has not lost yet*. The non-representability of $\mathfrak{CmAt}\mathfrak{A}$ amounts to that $\mathfrak{CmAt}\mathfrak{A} \notin \mathbf{SNr}_n \mathbf{CA}_{n+k}$ for some finite k because $\mathbf{RCA}_n = \bigcap_{k < \omega} \mathbf{SNr}_n \mathbf{CA}_{n+k}$. Can we **pin down** the value of k ? getting an estimate that is not ‘infinitely’ loose. *By adjusting the number of greens in the proof of theorem 9.4 to be the least possible that outfits the n ‘reds’, namely, $n+1$, one gets a finer result than Hodkinson’s [24] where Hodkinson uses an ‘overkill’ of infinitely many greens. By truncating the greens to be $n+1$, which is the smallest number $\lambda(> n)$ (of greens) for which \forall has a winning strategy in the private Ehrenfeucht–Fraïssé forth game $\mathbf{EF}_\lambda(\lambda, n)$, lifted to the rainbow game $G_\omega^{n+3}(\mathbf{AtCA}_{n+1, n})$, we could tell when $\mathbf{SNr}_n \mathbf{CA}_{n+k}$, $3 \leq k \leq \omega$ ‘stops to be atom-canonical’. This happened when $\mathfrak{CmAt}\mathfrak{A}$ ‘stopped to be representable at $k = 3$ ’. Here the number of nodes, namely, $n+3$ used by \forall in the graph game tells us that \mathfrak{CmAt} stopped to have m -dilations for any $m \geq n+3$. The*

finite number of rounds r gives no additional information, as far as non-atom canonicity is concerned, so we can save ourselves the hassle of finding the least such r . This is simply immaterial in the present context for here it is the (finite) number of nodes used by \forall during the play, which is $n + 3$, not the (finite) number of rounds $< \omega$ needed for him to implement his winning strategy, that matters to conclude that $\mathbf{SNr}_n\mathbf{CA}_{n+3}$ is not atom-canonical.

Now what if in the ‘Monk construction’ based on a graph having finite chromatic number as given in the construction in theorem 9.14, we replace this graph by \mathfrak{G} for which $\chi(\mathfrak{G}) = \infty$? Let us approach the problem abstractly. We proceed like in [21, §6.3] using the notation in *op.cit.* Let \mathfrak{G} be any graph. One can define a family of first order structures (labelled graphs) in the signature $\mathfrak{G} \times n$, denote it by $I(\mathfrak{G})$ like we did in the proof of theorem 9.14 as follows: The first order model M is in $I(\mathfrak{G}) \iff$ for all $a, b \in M$, there is a unique $p \in \mathfrak{G} \times n$, such that $(a, b) \in p$. If $M \models (a, i)(x, y) \wedge (b, j)(y, z) \wedge (c, l)(x, z)$, then $|\{i, j, l\}| > 1$, or $a, b, c \in \mathfrak{G}$ and $\{a, b, c\}$ has at least one edge of \mathfrak{G} . For any graph Γ , let $\rho(\Gamma)$ be the atom structure defined from the class of models satisfying the above, these are maps from $n \rightarrow M$, $M \in I(\Gamma)$, endowed with an obvious equivalence relation, with cylindrifiers and diagonal elements defined as Hirsch and Hodkinson define atom structures from classes of models, and let $\mathfrak{M}(\Gamma)$ be the complex algebra of this atom structure as defined on p.78 right before [21, Lemma 3.6.4].

We used the Monk-like algebras in [36, 7] to construct a sequence of bad Monk algebras converging to a good one proving non finite axiomatizability of \mathbf{RRA} and \mathbf{RCA}_n . Let us reverse the process, dealing with good Monk algebras converging to a bad one aspiring to prove non-first order definability of the class of strongly representable atom structures of \mathbf{RA} and \mathbf{RCA}_n .

Let \mathfrak{G} be a graph. Consider the relation algebra atom structure $\alpha(\mathfrak{G})$ defined exactly like in the proof of theorem 9.14. Then the relation complex algebra based on this atom structure will have an n -dimensional cylindric basis and, in fact, the cylindric atom structure of $\mathfrak{M}(\mathfrak{G})$ is isomorphic (as a cylindric algebra atom structure) to the atom structure $\mathbf{Mat}_n(\alpha(\mathfrak{G}))$ of all n -dimensional basic matrices over the relation algebra atom structure $\alpha(\mathfrak{G})$. It is plausible that one can prove that $\alpha(\mathfrak{G})$ is strongly representable $\iff \mathfrak{M}(\mathfrak{G})$ is representable $\iff \mathfrak{G}$ has infinite chromatic number, so that one gets the result, that the class of strongly representable atom structure both \mathbf{RAs} and \mathbf{CAs} of finite dimension at least three, is not elementary in one go. The underlying idea here is that shade of red ρ will appear in the *ultrafilter extension* of \mathfrak{G} , if it has infinite chromatic number, as a reflexive node [21, Definition 3.6.5] and its n -copies (ρ, i) , $i < n$, can be used to completely represent $\mathfrak{M}(\mathfrak{G})^+$.

For a variety \mathbf{V} of Boolean algebras with operators, let $\mathbf{Str}(\mathbf{V})$ be the following class of atom structures: $\{\mathfrak{F} : \mathfrak{Cm}\mathfrak{F} \in \mathbf{V}\}$ [19, Definition 2.89]. Using a rainbow construction, we showed that $\mathbf{SNr}_n\mathbf{CA}_{n+k}$ is not atom-canonical, for any $k \geq 3$. We know that $\mathbf{Str}(\mathbf{RCA}_n)$ is not elementary [21, Corollary 3.7.2]; this is proved using Monk-like algebras. Fix finite $k > 2$. Let $\mathbf{V}_k = \mathbf{SNr}_n\mathbf{CA}_{n+k}$. Then $\mathbf{Str}(\mathbf{SNr}_n\mathbf{CA}_{n+k})$ is not elementary $\implies \mathbf{V}_k$ is not-atom canonical [19, Theorem 2.84]. *But the converse implication, namely, \mathbf{V}_k is not atom-canonical $\implies \mathbf{Str}(\mathbf{V}_k)$ not elementary does not hold in general.* It is easy to show that there has to be a finite $k < \omega$ such that $\mathbf{Str}(\mathbf{V}_k)$ is not elementary. It therefore seems plausible that Rainbows and Monk-like algebras can be reconciled to control the value of m for which $\mathbf{Str}(\mathbf{SNr}_n\mathbf{CA}_{n+m})$ ($m > 2$) is not elementary, thereby

using a combination of variants of Ramsey’s theorem and Erdős probabalistic graphs with a ‘rainbow intervention’ to control the ‘extra dimensions’. The latter probabalistic technique is used in the aforementioned [19, Corollary 3.7.2].

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